Representations of topological groups and Kac-Moody groups

by

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Contents

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Declarations

I hereby declare that the work presented in this thesis is original and entirely my own, unless a collaboration has been specified. Whenever known material is used, it is clearly referenced in the text. The original material is a compilation of three research papers. Chapter 3 is taken from a paper by myself entitled "Frobenius Reciprocity for Topological Groups", which is soon to appear in the journal *Communications in Algebra* [33]. Chapter 4, Chapter 5, Chapter 6 and Section 7.4 of Chapter 7 are adapted from a joint paper between Dmitriy Rumynin and myself, entitled "Kac-Moody Groups and Cosheaves on Davis building", which is published in *Journal of Algebra* [32]. Section 7.3 of Chapter 7 is a joint paper between Inna Capdeboscq, Dmitriy Rumynin and myself, which is currently available as a preprint [10].

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

This thesis studies continuous representations of topological groups over a field $\mathbb{F}$. We form three categories of continuous representations of such a group G: the category of discrete representations $\mathcal{M}_d(G)$, category of linearly topologized complete representations $\mathcal{M}_{ltc}(G)$ and the category of (linearly) compact representations $\mathcal{M}_c(G)$. We obtain a version of Frobenius reciprocity in these categories. We then study categories $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$ of discrete A-semisimple smooth representations of a locally compact totally disconnected group G, and A-semisimple smooth representations of G coming from a fixed character χ, where $A \leqslant G$ is a closed central subgroup. We establish an equivalence between the categories above and certain subcategories of the category of smooth modules over the Hecke algebra of G. Our main results encompass an upper bound on the projective dimension of $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$, as well as construction of explicit projective resolutions of objects in these categories, in the case when G acts continuously on a simplicial set $\mathcal{X}_\bullet$ with a contractible geometric realisation $|\mathcal{X}|$. We also study categories of G-equivariant sheaves and cosheaves on the simplicial set $\mathcal{X}_\bullet$. We prove a localisation result relating those to the category of smooth representations. We also prove existence of finitely generated projective resolutions of Schneider-Stuhler type when $|\mathcal{X}|$ is a tree. We finish with an investigation of complete locally compact totally disconnected Kac-Moody groups. We define a simplicial complex with a contractible geometric realisation on which they act. We conclude with a study of cocompact lattices in locally pro-p-complete Kac-Moody groups.

Notation

Throughout this thesis we use the following notations:

- group elements are in bold letters: $\mathbf{g}, \mathbf{h}, \mathbf{k}, \ldots$, etc.;
- elements of fields, rings, vector spaces and sets are in standard font letters;
- G is a topological group in Chapters 2, 3, a locally compact totally disconnected group in Chapters 4, 5, 6, and a group with a (B, N)-pair or a minimal Kac-Moody group in Chapter 7;
- $H \leqslant G$ is a closed subgroup of the topological group G in Chapter 3;
- $A \leqslant G$ is a closed central subgroup of the locally compacted totally disconnected group G (Chapters 4, 5, 6);
- R is an associative unital ring;
- $\mathbb{F}$ is a field of arbitrary characteristic, but in Chapters 4, 5, 6, 7 conditions on the characteristic of $\mathbb{F}$ will be imposed;
- $\widetilde{\mathbb{F}}$ is a field which is an extension of $\mathbb{F}$, such that as an $\mathbb{F}$-algebra it is generated by the image of a character $\chi : A \to \widetilde{\mathbb{F}}^{\times}$;
- $\widetilde{\mathbb{F}}_{\chi}$ is the representation of A obtained from a field extension $\widetilde{\mathbb{F}} \supseteqq \mathbb{F}$, where $\widetilde{\mathbb{F}}$ is generated as an $\mathbb{F}$-algebra by the image of a character $\chi : A \to \widetilde{\mathbb{F}}^{\times}$;
- $\mathbb{K}$ is a non-archimedean local field in Chapters 4, 5, 6 and in Chapter 7 it is an arbitrary field over which the minimal Kac-Moody groups are defined;

- $\mathbb{F}_q$ is the finite field of q elements, where $q = p^a$, for some prime p;
- $\mathbb{Q}_p$ is the field of p-adic numbers, where p is a prime;
- $\mathbb{Z}_p$ is the ring of p-adic integers, where p is a prime;
- $\mathbb{C}$ is the field of complex numbers;
- $\mathbb{R}$ is the field of real numbers;
- $\mathcal{M}(G)$ is the category of continuous representations of a topological group G in Chapter 2 and Chapter 3, and the category of smooth representations of a locally compact totally disconnected group G in Chapters 4, 5, 6, 7;
- $\mathcal{M}_d(G)$ is the category of discrete representations of a topological group G;
- $\mathcal{M}_{ltc}(G)$ is the category of linearly topologized complete representations of a topological group G;
- $\mathcal{M}_c(G)$ is the category of (linearly) compact representations of a topological group G;
- $\mathcal{M}_A(G)$ is the category of smooth representations of a locally compact totally disconnected group G, which are semisimple as representations of the closed central subgroup $A \leqslant G$;
- $\mathcal{M}_{A,\chi}(G)$ is the full subcategory of $\mathcal{M}_A(G)$ consisting of representations which are direct sums of $\widetilde{\mathbb{F}}_\chi$, for $\chi \in \mathrm{Irr}(\mathbb{F}A)$;
- $\mathrm{Irr}(\mathbb{F}A)$ is the set of all characters $\chi : A \to \widetilde{\mathbb{F}}^\times$, such that $\widetilde{\mathbb{F}}$ is generated as an $\mathbb{F}$-algebra by the image of χ;
- Res_H^G is the restriction functor from a continuous representation of G to a continuous representation of a subgroup $H \leqslant G$;
- Ind_H^G is the induction functor, which is right adjoint to Res_H^G. It is a functor from the category of continuous representations of a subgroup $H \leqslant G$ to the category of continuous representations of G;

- Coind_H^G is the coinduction functor, which is left adjoint to Res_H^G. It is a functor from the category of continuous representations of a subgroup $H \leqslant G$ to the category of continuous representations of G. In the case when G is locally compact totally disconnected (i.e., Chapters 4, 5, 6) Coind_H^G is denoted $c - \mathrm{Ind}_H^G$ and is called compact induction;

- $a{-}\mathrm{Ind}_H^G$ is algebraic induction, i.e., for a subgroup $H \leqslant G$ and $(\sigma, W) \in \mathcal{M}(H)$, $a - \mathrm{Ind}_H^G(\sigma) = \mathbb{F}G \otimes_{\mathbb{F}H} W$;

- μ is a (left) Haar measure on a locally compact group totally disconnected group G;

- μ_K is the normalised at a compact subgroup $K \leqslant G$ (left) Haar measure on a locally disconnected group G;

- $\mathcal{H}(G, \mathbb{F}, \mu_K)$ is the Hecke algebra of the locally compact totally disconnected group G over the field $\mathbb{F}$, where the field $\mathbb{F}$ is assumed to be K-ordinary and $K \leqslant G$ is a compact subgroup;

- $\mathcal{M}(\mathcal{H})$ is the category of smooth (non-degenerate) modules over the Hecke algebra of a locally compact totally disconnected group G;

- $\mathcal{M}_A(\mathcal{H})$ is the subcategory of $\mathcal{M}(\mathcal{H})$ obtained by taking the image of $\mathcal{M}_A(G)$ under the functor $\mathcal{F}$ in Theorem 4.5.2;

- $\mathcal{M}_{A,\chi}(\mathcal{H})$ is the subcategory of $\mathcal{M}(\mathcal{H})$ as defined in Section 5.1;

- $\mathcal{M}_A(G)^U$ is the subcategory of $\mathcal{M}_A(G)$ with objects those smooth representations generated by their U-fixed vectors;

- $\mathcal{M}_A(G)^\circ$ is the union of various $\mathcal{M}_A(G)^U$;

- $\mathcal{X}_\bullet$ is a simplicial set;

- $\mathcal{BT}$ is the Bruhat-Tits building of an algebraic group or the building of a group with a (B, N)-pair structure;

- $\mathcal{BT}_\bullet$ is the Bruhat-Tits building as above, but viewed as a simplicial set;
- $\mathfrak{B}$ is just a building;
- $\mathcal{D}_\bullet$ is the Davis building of a group with a generalised (B,N)-pair or a Kac-Moody group;
- $\mathfrak{D}$ is a root datum of type $\mathcal{A}$, where $\mathcal{A}$ is a generalised Cartan matrix;
- $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ is the category of G-equivariant cosheaves on $\mathcal{X}_\bullet$;
- $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)$ is the category of discrete G-equivariant cosheaves on $\mathcal{X}_\bullet$;
- $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)$ is the category of A-semisimple G-equivariant cosheaves on $\mathcal{X}_\bullet$;
- $\mathrm{Csh}_{G,A,\chi}(\mathcal{X}_\bullet)$ is the category of A-semisimple G-equivariant cosheaves on $\mathcal{X}_\bullet$ with a fixed character χ;
- $\mathcal{G}$ is a system of subgroups of the locally compact totally disconnected group G;
- $\underset{\sim}{V}$ is the trivial cosheaf on $\mathcal{X}_\bullet$;
- $\underset{\sim}{V^{\mathcal{G}_x}}$ is the cosheaf of invariants of a contravariant system of subgroups $\mathcal{G}$ of G on the simplicial set $\mathcal{X}_\bullet$;
- $\mathcal{A}[\Sigma^{-1}]$ is the localisation of the category $\mathcal{A}$ at the set of morphisms Σ;
- $\Sigma(W,S)$ is the Coxeter complex of a Coxeter system (W,S);
- $\Gamma(W,S)$ is the presentation diagram of a Coxeter system (W,S);
- $\overline{G}$ in Section 7.3 is the Caprace-Rémy completion of a minimal Kac-Moody group G;
- $\widehat{G}$ is the local pro-p completion of a minimal Kac-Moody group G;
- $\widetilde{G}$ is the symbol we use if we want to talk in parallel about the completions of a minimal Kac-Moody group G that give a locally compact totally disconnected group.

Chapter 1

Introduction

Topological groups are groups endowed with a topology, such that multiplication and inversion in the group are continuous maps in this topology. They have been a subject of research for decades not only because of their rich structure, but also because of their representation theory. The representation theory of compact groups had been studied extensively and is well-understood. Naturally, the interest sprang to locally compact groups too. Every locally compact group G with $G^0 \leqslant G$ the connected component of the identity, fits into the exact sequence

$$1 \to G^0 \to G \to G/G^0 \to 1,$$

where the quotient G/G^0 is locally compact totally disconnected. Thus, morally, to understand locally compact groups, we must also understand their locally compact totally disconnected cousins. However, these remained unstudied for a long time. It was not until 1994 when a groundbreaking structural result of George Willis finally drew the interest of the mathematical community to the locally compact totally disconnected case [57]. Ever since these groups have been widely studied. Examples of locally compact totally disconnected groups include algebraic groups over non-archimedean local fields with discrete valuations and consequently p-adic groups. The representation theory of all such groups is very rich and of great interest to both representation and number theorists since these groups fit into the Langlands programme [9], [46]. Taking into account the topology of the groups, one can induce a continuity condition on the representations and thus obtain the so-called smooth representations. Simple smooth representations are very important, as they lie in the heart of the Langlands programme [52], [9].

Another trendy example of locally compact totally disconnected groups comes from completions of Kac-Moody groups over finite fields. In general, Kac-Moody groups can be thought of as generalisations of reductive algebraic groups, as they arise from certain infinite-dimensional Lie algebras - the Kac-Moody algebras. However, Kac-Moody groups can be studied separately from their Lie algebras. More precisely, to any generalised Cartan matrix $\mathcal{A}$, root datum $\mathfrak{D}$ and a field $\mathbb{K}$, one can associate a Kac-Moody group purely combinatorically - by writing down generators and relations. This presentation is due to Tits [53], and Carter-Chen [17]. The resulting group is called a *minimal Kac-Moody group* over the field $\mathbb{K}$, denoted $G_{\mathfrak{D}}(\mathbb{K})$. One can endow $G_{\mathfrak{D}}(\mathbb{K})$ with various topologies to make it into a topological group. We can take a completion in the chosen topology. This results in a *complete Kac-Moody group.* When the ground field $\mathbb{K}$ is finite the complete groups can be locally compact totally disconnected.

The main topic of this thesis is studying the representation theory of locally compact totally disconnected groups. The first problem we address is extending a well-known result from the representation theory of finite groups - Frobenius reciprocity, to the setting of an arbitrary topological group. Recall that for a subgroup $H \leqslant G$, every representation of G gives rise to a representation of H by simply restricting the action of G to the subgroup H. Conversely, for every representation of H we can obtain a representation of G by the process of induction. Induction and restriction give a pair of adjoint functors between the categories of representations of G and representations of H. This adjunction is known as *Frobenius reciprocity.* We investigate when such a pair of adjoint functors exists in the case of a topological group G and a closed subgroup $H \leqslant G$. By adding a continuity condition to the standard definition of a representation, we take into account the fact that there is a topology on G. What we obtain is a *continuous representation.* In the special case when we look at the category of discrete continuous representations of a locally compact totally disconnected group, we obtain the category of smooth representations.

The second main problem we address is studying projective resolutions of smooth representations of a locally compact totally disconnected group which acts on a simplicial set. Let us explain our motivation for addresing this problem. On one side, there is the work of Bernstein on p-adic algebraic groups, appearing in his unpublished lecture notes from a course in Harvard in 1992 [4]. He gives a finite bound on the projective dimension of the category of smooth representations of a p-adic group G by using its Bruhat-Tits building [4]. This is a simplicial complex $\mathcal{BT}$ on which the group acts. It has many nice properties - for example, its dimension is the rank of the group, and its geometric realisation is a contractible space [5]. Bruhat-Tits buildings have not only been attractive due to their topological properties, but they have proved to be incredibly useful when studying representations of reductive groups. In their seminal paper, Schneider and Stuhler use the Bruhat-Tits building of a connected reductive algebraic group G over a non-archimedean local field to construct projective resolutions of admissible smooth representations [52]. They do this by passing from the category of smooth representations of G to a category of equivariant objects on $\mathcal{BT}$, called *coefficient systems* on $\mathcal{BT}$. The explicitness of the construction proves to be quite fruitful - it leads to a proof of a conjecture by Kazhdan for a formula for the orthogonality of Harish-Chandra characters [52], [37]. It is worth noting, that a similar approach with constructing sheaves on the building was taken by Ronan and Smith in order to study the modular representation theory of Chevalley groups [48].

Now let us move back to our setting. As mentioned earlier, reductive groups are locally compact totally disconnected. Thus, inspired by the work of both Bernstein and Schneider-Stuhler, we would like to generalise their constructs to the case of an arbitrary locally compact totally disconnected group acting on a simplicial set $\mathcal{X}_\bullet$. As a main example we have complete Kac-Moody groups over finite fields. Moreover, we explicitly construct the simplicial set on which these groups act. Another attractive property of locally compact groups is that they admit a left invariant measure - the Haar measure. Going on a slight tangent from our representation theory investigation we look at cocompact lattices of minimal covolume in two specific completions of rank 2 Kac-Moody groups. This is the final result we present in this work.

Let us give a chapter-by-chapter outline of this thesis. Chapter 2 gives basic definitions and examples of topological groups and their continuous representations. Chapter 3 is our study of Frobenius reciprocity for topological groups. We consider three specific categories of continuous representations - the category of discrete representations $\mathcal{M}_d(G)$, the category of linearly topologized and complete representations $\mathcal{M}_{ltc}(G)$ and the category of (linearly) compact representations $\mathcal{M}_c(G)$. All representations we study in the chapter are over an associative unital ring R. We establish the following result:

Theorem (Main Theorem 1)**.** *(Theorem 3.3.4, Lemma 3.3.7, Theorem 3.4.5, Theorem 3.5.7) Let G be a topological group and $H \leqslant G$ a closed subgroup. The restriction functor*

$$\mathrm{Res}_H^G : \mathcal{M}_\star(G) \to \mathcal{M}_\star(H)$$

has the following properties:

1. *In $\mathcal{M}_d(G)$ the functor Res_H^G always has a right adjoint, given by the induction functor Ind_H^G, and has a left adjoint Coind_H^G if H is also open.*

2. *In $\mathcal{M}_{ltc}(G)$ and $\mathcal{M}_c(G)$ the functor Res_H^G always has a left adjoint Coind_H^G and has a right adjoint Ind_H^G if H is also open.*

In Chapter 4 we specialise to studying the category $\mathcal{M}(G)$ of continuous discrete representations of locally compact totally disconnected topological groups, i.e., smooth representations, over a field $\mathbb{F}$. At the beginning the field $\mathbb{F}$ is arbitrary, but eventually we put restrictions on its characteristic in order to obtain a well-defined Hecke algebra of G over $\mathbb{F}$. Now let G be a locally compact totally disconnected group. Except for $\mathcal{M}(G)$, there are two further categories of interest: $\mathcal{M}_A(G)$ -

the category of A-semisimple representations of G, where $A \leqslant G$ is a closed central subgroup, and $\mathcal{M}_{A,\chi}(G)$ - the category of A-semisimple smooth representations of G coming from a fixed character χ of A. Next we review the construction of the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of a locally compact totally disconnected group, where $K \leqslant G$ is a compact subgroup and μ_K is the normalised at K (left) Haar measure of G. The Hecke algebra is defined over the field $\mathbb{F}$. For this to hold, we need to put restrictions on the characteristic of $\mathbb{F}$ - in particular, we need the order $|K|$ of K to be invertible in $\mathbb{F}$. We keep these restrictions for the whole Chapter 4 and 5. In Theorem 4.5.2 and Corollary 4.5.3 we establish an equivalence between the categories above and suitable subcategories of $\mathcal{M}(\mathcal{H})$ - the category of smooth modules over the Hecke algebra. These equivalences are crucial for our later results on projective resolutions. The reason for this is that the property of having enough projectives in a category is established on the Hecke algebra side, and we need the above equivalences to obtain this property for $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$.

In Chapter 5 we concentrate on projective resolutions in $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$. We study the projective dimension of these categories. In the case when there exists a continuous action of the group G on a simplicial set $\mathcal{X}_\bullet$, we obtain the following bound on the projective dimension of $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$:

Theorem (Main Theorem 2)**.** *(Theorem 5.3.1) Let G be a locally compact totally disconnected group, A its closed central subgroup, and $K \leqslant G$ a compact subgroup, such that the field $\mathbb{F}$ is K-ordinary. Suppose G acts continuously on an n-dimensional simplicial set $\mathcal{X}_\bullet$ with contractible geometric realisation $|\mathcal{X}|$, so that A acts trivially on $\mathcal{X}_\bullet$. Suppose that the action of G extends to $|\mathcal{X}|$ (as in Proposition 5.2.6). Suppose further that the stabiliser G_x of any non-degenerate simplex $x \in \mathcal{X}_{(k)}$ is not only open (that follows from continuity) but also compact modulo A. If the field $\mathbb{F}$ is G_x/A-ordinary for any $x \in \mathcal{X}_k$, then*

$$\operatorname{proj.dim}(\mathcal{M}_{A,\chi}(G)) \leqslant n \quad \text{and} \quad \operatorname{proj.dim}(\mathcal{M}_A(G)) \leqslant n.$$

In Chapter 6 we are again in the setting of a locally compact group G acting continuously on a simplicial set $\mathcal{X}_\bullet$. We start by defining G-equivariant cosheaves and G-equivariant sheaves on $\mathcal{X}_\bullet$. These objects form abelian categories which we respectively denote by $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ and $\mathrm{Sh}_G(\mathcal{X}_\bullet)$. Our first main result is a localisation theorem:

Theorem (Localisation Theorem). *(Theorem 6.2.5) Consider a continuous action of the locally compact totally disconnected group G on a simplicial set $\mathcal{X}_\bullet$, where the central subgroup A acts trivially. If $|\mathcal{X}|$ is connected, then there are equivalences of categories:*

$$\mathfrak{H}[\Sigma^{-1}] : \mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}] \to \mathcal{M}(G)$$

$$\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma_A^{-1}] \xrightarrow{\cong} \mathcal{M}_A(G) \text{ and } \mathrm{Csh}_{G,A,\chi}(\mathcal{X}_\bullet)[\Sigma_{A,\chi}^{-1}] \xrightarrow{\cong} \mathcal{M}_{A,\chi}(G),$$

where Σ_A and $\Sigma_{A,\chi}$ are intersections of Σ with the corresponding subcategories.

Note that to establish this theorem we do not use the existence of the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ as defined in Section 4.4 and, thus, there are no restrictions on $\mathbb{F}$ for this result. We then move on to *Schneider-Stuhler resolutions*. These are resolutions in $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$ by finitely generated projective objects inspired by Schneider-Stuhler's construction for connected reductive algebraic groups. We prove the existence of a Schneider-Stuhler resolution in the case when the locally compact totally disconnected group G acts on a tree (Theorem 6.3.8). We conjecture that the result carries over to higher dimensions, however, we do not prove this conjecture (Conjecture 6.3.7).

In the final Chapter 7 we tackle Kac-Moody groups. We begin by introducing groups which allow a (B, N)-pair structure and their associated buildings. We then define minimal Kac-Moody groups - i.e., the Kac-Moody groups without any topology. These groups can be defined for any generalised Cartan matrix $\mathcal{A}$, root datum $\mathfrak{D}$ of type $\mathcal{A}$ and a field $\mathbb{K}$. Here the field $\mathbb{K}$ is completely arbitrary, we do not put any restrictions on it. We next specialise our attention to the case when $\mathbb{K} = \mathbb{F}_q$, the finite field of $q = p^a$ elements, where p is a prime. We define the relevant for our investigation complete Kac-Moody groups in this setting - these are the Kac-Moody groups which are locally compact totally disconnected. In Section

7.3 we investigate cocompact lattices in complete rank 2 Kac-Moody groups. The reason we restrict our attention to rank 2 is that in rank $n > 2$ Kac-Moody groups as a whole there are no cocompact lattices, except in some special cases which have already been established [14]. We concentrate our investigation on cocompact lattices in locally pro-p-complete rank 2 Kac-Moody groups $\widehat{G}$ by relating them to the already classified by Capdeboscq and Thomas edge-transitive cocompact lattices in the Caprace-Rémy complete groups $\overline{G}$ [12]. This classification relies on certain properties of the p-elements in the complete Kac-Moody group. We call a group satisfying the required properties *p-well-behaved*. Our first main theorem shows that the locally pro-p-complete Kac-Moody groups are p-well-behaved (Theorem 7.3.11 and Theorem 7.3.5). We then establish pushing and pulling procedures of lattices between the two groups and show that the covolume of a lattice does not change when performing these procedures. We thus arrive at our main theorem:

Theorem (Main Theorem 3)**.** *(Theorem 7.3.19) Let $\mathcal{A}$ be a symmetric 2×2 generalised Cartan matrix with all $|a_{ij}| \geqslant 2$. Let $\mathfrak{D}$ be a simply-connected root datum of type $\mathcal{A}$. The following statements hold for the corresponding (to $\mathfrak{D}$) locally pro-p-complete Kac-Moody group $\widehat{G}$ over the field of $q = p^a$ elements:*

1. *$\widehat{G}$ admits a cocompact lattice.*

2. *If $q \geqslant 514$, then there exist $\delta \in \{1, 2, 4\}$, such that*

$$\min\{\widehat{\mu}(\Gamma \backslash \widehat{G}) \mid \Gamma \textit{ is a cocompact lattice}\} = \frac{2}{(q+1)|Z(G)|\delta} .$$

We finish by defining a class of groups similar to complete Kac-Moody groups - *topological groups of Kac-Moody type*. These are locally compact totally disconnected groups with a generalised (B, N)-pair structure, which satisfies certain properties (Section 7.4.2). We show that there exists a simplicial complex with a contractible geometric realisation on which the topological groups of Kac-Moody type, as well as the complete Kac-Moody groups, act. We call it the *Davis building*. We are in a situation to apply our results for projective resolutions from Chapter 5 and

Chapter 6 to Kac-Moody groups. The field $\mathbb{F}$ from Chapter 4 and 5 appears again. We put restrictions on its characteristic in order to obtain our Hecke algebra over the field $\mathbb{F}$. Our final results are:

Corollary (Main Corollary 1)**.** *(Corollary 7.4.13) Let G be a topological group of Kac-Moody type, A its central closed subgroup, such that B/A is compact. The localisation functor for the category of A-semisimple G-representations over a field $\mathbb{F}$*

$$\mathcal{M}_A(G) \xrightarrow{\cong} \mathrm{Csh}_{G,A}(\mathcal{D}_\bullet)[\Sigma_A^{-1}]$$

is an equivalence of categories. Let $C \leqslant G$ be a compact subgroup, such that the field $\mathbb{F}$ is C-ordinary. Suppose further that $\mathbb{F}$ is G_x/A-ordinary for any $x \in \mathcal{D}_\bullet$, where $\mathcal{D}_\bullet$ is the Davis building of G. Then

$$\mathrm{proj.\,dim}(\mathcal{M}_A(G)) \leqslant \sup_{J \in \mathcal{S}ph(S)} |J|$$

where $|J|$ denotes the cardinality of J.

Corollary (Main Corollary 2)**.** *(Corollary 7.4.18) Let $\widetilde{G}$ be a complete Kac-Moody group over a finite field $\mathbb{F}_q$, such that $\widetilde{G}$ is locally compact totally disconnected. Let $C \leqslant \widetilde{G}$ be a compact subgroup, such that the field $\mathbb{F}$ is C-ordinary. Let $\mathcal{D}_\bullet$ be the Davis building of $\widetilde{G}$. If the field $\mathbb{F}$ is also $\widetilde{G}_x/K$-ordinary for any $x \in \mathcal{D}_\bullet$, then*

$$\mathrm{proj.\,dim}(\mathcal{M}(\widetilde{G})) \leqslant \dim(\mathcal{D}_\bullet) \textit{ and } \mathcal{M}(\widetilde{G}) \cong Csh_{\widetilde{G}}(\mathcal{D}_\bullet)[\Sigma^{-1}].$$

Moreover, for A its central closed subgroup, such that $\widetilde{B}/A$ is compact, we have the equivalence

$$\mathcal{M}_A(\widetilde{G}) \xrightarrow{\cong} \mathrm{Csh}_{G,A}(\mathcal{D}_\bullet)[\Sigma_A^{-1}].$$

If the field $\mathbb{F}$ is $\widetilde{G}_x/A$-ordinary for any $x \in \mathcal{D}_\bullet$, then

$$\mathrm{proj.\,dim}(\mathcal{M}_A(\widetilde{G})) \leqslant \sup_{J \in \mathcal{S}ph(S)} |J|$$

where $|J|$ denotes the cardinality of J.

Chapter 2

Representations of topological groups

We start by introducing our basic objects of study. The majority of this chapter consists of well-known definitions. The main reference is Warner [56]. The examples at the end are taken from a paper by the author of this thesis [33].

Defintion 2.0.1. A *topological group* G is a pair (G, τ_G), where G is a group and τ_G is a topology on the underlying set of G, such that the maps

$$m : G \times G \to G, \ (\mathbf{g}, \mathbf{h}) \mapsto \mathbf{gh},$$

and

$$i : G \to G, \ \mathbf{g} \mapsto \mathbf{g}^{-1}$$

are continuous with respect to τ_G. Note that the topology on $G \times G$ is the product topology induced by τ_G.

A *homomorphism of topological groups* is a group homomorphism $\varphi : G \to H$, which is also a continuous map of topological spaces.

Similarly, for an associative unital ring R, we can define a topological R-module as follows:

Defintion 2.0.2. [56] A *topological R-module* V is a pair $(V, \mathcal{T}_V)$, where V is an

R-module and $\mathcal{T}_V$ is a topology on V, such that $((V,+),\mathcal{T}_V)$ is a topological group and the R-action map $\cdot : R \times V \to V, (r,v) \mapsto r \cdot v$ is continuous with respect to $\mathcal{T}_V$ on the right and the product topology on the left (R is endowed with the discrete topology).

A *homomorphism of topological R-modules* is an R-module homomorphism $f : V_1 \to V_2$, which is also a continuous map with respect to the topologies $\mathcal{T}_{V_1}$ and $\mathcal{T}_{V_2}$.

Throughout, by an R-module we always mean a left R-module. If we require a right R-module we would specify this. All results are proved for left modules, however, they remain true for right modules.

We are ready to define our main object of study:

Defintion 2.0.3. [33] Let R be an associative ring with 1 and G a topological group. A *continuous representation* of G is a pair (π, V), such that:

1. (representation) $(V, \mathcal{T}_V)$ is a topological R-module and $\pi : G \to \mathrm{Aut}_R(V)$ a homomorphism,

2. (continuity) The map $\phi : G \times V \to V$, defined by $(\mathbf{g}, v) \mapsto \pi(\mathbf{g})v$, is continuous with respect to the product topology on the left and $\mathcal{T}_V$ on the right.

We sometimes refer to representations of G as G-modules. Now, for a topological group G and an associative unital ring R, we can form a *category of continuous representations of* G, denoted $\mathcal{M}(G)$, as follows:

- The objects $Ob(\mathcal{M}(G))$ are continuous representations of G, i.e., pairs (π, V) as above;

- For $(\pi_1, V_1), (\pi_2, V_2) \in Ob(\mathcal{M}(G))$ a morphism $f \in \mathrm{Hom}_{\mathcal{M}(G)}(V_1, V_2)$ is given by a homomorphism of topological R-modules $f : V_1 \to V_2$, such that $f(\pi_1(\mathbf{g})v) = \pi_2(\mathbf{g})f(v)$, for all $\mathbf{g} \in G$ and all $v \in V_1$. We refer to the second part of this condition as G-linearity.

It is clear that the continuity condition of Definition 2.0.3 depends on the topology of V. Let us look at some examples of continuous representations, where a specific topology on the R-modules is chosen.

Example 2.0.4. (cf. [33]) Let (π, V) be a continuous representation of a topological group G, such that the topology $\mathcal{T}_V$ is discrete. We call such a representation a *discrete representation* of G and denote the category of all discrete representations of G by $\mathcal{M}_d(G)$. As all maps in the discrete topology are continuous, the morphisms in $\mathcal{M}_d(G)$ are just R-module maps, which are also G-linear.

Example 2.0.5. (cf. [4], [9], [52]) In the special case when we consider discrete representations of a locally compact totally disconnected group G, the category of discrete representations is known in the literature as the category of *smooth representations* of G. We denote it $\mathcal{M}(G)$.

In the next two examples the topological R-modules $(V, \mathcal{T}_V)$ are given a *linear* topology. We say that a topology $\mathcal{T}_V$ is *linear*, or that V is *linearly topologized*, if the open R-submodules of V form a fundamental system of neighbourhoods at zero [56].

Example 2.0.6. [33] For a topological group G, we define the category of *linearly topologized representations* of G, denoted $\mathcal{M}_{ltc}(G)$. The objects are pairs (π, V), where (π, V) is a continuous representation of G with $(V, \mathcal{T}_V)$ being a linearly topologized R-module, such that V is a complete topological space with respect to $\mathcal{T}_V$. The morphisms are continuous R-module homomorphisms, which are also G-linear.

Example 2.0.7. [33] We also define the category of *linearly compact* representations of G, or just compact representations, denoted $\mathcal{M}_c(G)$. The objects are pairs (π, V), where (π, V) is a continuous representation of G and $(V, \mathcal{T}_V)$ is a linearly topologized complete R-module, such that for every open R-submodule $W \leqslant V$, V/W is an R-module of finite length. Such modules are called *linearly compact*. The morphisms are defined as in $\mathcal{M}_{ltc}(G)$.

Chapter 3

Frobenius reciprocity for topological groups

To the best of my knowledge, the material in this chapter, except in Section 3.2, is original. It appears in a paper by the author, entitled "Frobenius Reciprocity for Topological Groups", which is to appear in the journal *Communications in Algebra* [33]. Throughout the chapter, G denotes a topological group, $H \leqslant G$ a closed subgroup and R an associative ring with unity.

3.1 Restriction, Induction and Coinduction

Fix a topological group G and a closed subgroup $H \leqslant G$. Let (π, V) be a continuous representation of G over R. By restricting the map $\pi : G \to \mathrm{Aut}_R(V)$ to a map $\pi|_H : H \to \mathrm{Aut}_R(V)$ we obtain a representation of H. This yields a restriction of $\phi : G \times V \to V$, $\phi : (\mathbf{g}, v) \mapsto \pi(\mathbf{g})v$, to $\phi|_H : H \times V \to V$. As a restriction of a continuous map is continuous, the pair $(\pi|_H, V)$ is a continuous representation of H. Applying the same reasoning on morphisms, we obtain a functor

$$\mathrm{Res}_H^G : \mathcal{M}(G) \to \mathcal{M}(H),$$

$$(\pi, V) \mapsto (\pi|_H, V),\ (f : V_1 \to V_2) \mapsto (f : V_1 \mapsto V_2),$$

called the *restriction functor*.

Recall the following standard definition:

Defintion 3.1.1. (cf. [36, Definition 1.5.2], [41, IV.1, Definition]) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Two functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ are called *adjoint* if for every $X \in Ob(\mathcal{C})$ and $Y \in Ob(\mathcal{D})$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{G}(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), Y)$$

which is natural in both variables. We say that $\mathcal{F}$ is left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is right adjoint to $\mathcal{F}$.

To define induced representations, we would like to investigate functors adjoint to Res_H^G. However, in the generality of continuous representations of topological groups, such functors do not necessarily have to exist. We make the following definition:

Defintion 3.1.2. Let G be a topological group, $H \leqslant G$ a closed subgroup, and $\mathrm{Res}_H^G : \mathcal{M}(G) \to \mathcal{M}(H)$ the restriction functor. Suppose there exist functors

$$\mathrm{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G) \text{ and } \mathrm{Coind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G),$$

such that Ind_H^G is right adjoint to Res_H^G and Coind_H^G is left adjoint to Res_H^G. We call Ind_H^G the *induction functor*, and Coind_H^G the *coinduction functor*. We call the image of a continuous representation of H under Ind_H^G an *induced representation*, and its image under Coind_H^G a *coinduced representation*.

In the case of abstract groups, the adjunction relation between restriction and induction, as well as restriction and coinduction, is known as Frobenius reciprocity. We keep the terminology the same in our setting and thus we have:

Defintion 3.1.3. Let G be a topological group and $H \leqslant G$ a closed subgroup. Suppose the functors Ind_H^G and Coind_H^G exist. Then the natural isomorphisms

$$\mathrm{Hom}_{\mathcal{M}_d(G)}\left((\pi, V), \mathrm{Ind}_H^G((\sigma, W))\right) \cong \mathrm{Hom}_{\mathcal{M}_d(H)}\left(\mathrm{Res}_H^G((\pi, V)), (\sigma, W)\right)$$

and

$$\operatorname{Hom}_{\mathcal{M}_d(G)}\big(\operatorname{Coind}_H^G((\sigma, W)), (\pi, V)\big) \cong \operatorname{Hom}_{\mathcal{M}_d(H)}\big((\sigma, W), \operatorname{Res}_H^G((\pi, V))\big),$$

for $(\pi, V) \in \mathcal{M}_d(G)$ and $(\sigma, W) \in \mathcal{M}_d(H)$, are called *Frobenius reciprocity for topological groups.*

3.2 Existence of adjoints and the Freyd Adjoint Functor Theorem

As noted in the previous section, functors adjoint to a given functor do not necessarily exist. However, there is a useful criterion for existence of such functors, called the Freyd Adjoint Functor Theorem. We would like to use this to establish conditions for existence of induced and coinduced representations. All the material in this section is known, and the reference is MacLane's book "Categories for the working mathematician"[41].

Let us start with some standard definitions. A category $\mathcal{C}$ is called *small* if both the collection of objects and morphisms are sets. A category $\mathcal{C}$ is called *locally small* if for every pair of objects $A, B \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is a set. A *small diagram* in $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ is a small category. A *small limit* in $\mathcal{C}$ (respectively *small colimit*) is the limit (respectively colimit) of a small diagram. More precisely, for a diagram $F : \mathcal{J} \to \mathcal{C}$, the limit of F is a pair (L, f), where L is an object of $\mathcal{C}$ and $f = \{f_i\}$, where $f_i : L \to F(i)$, is a family of morphisms in $\mathcal{C}$, indexed by the objects i of $\mathcal{J}$, such that for every morphism $\phi : i \to j$ in $\mathcal{J}$, we have $F(\phi) \circ f_i = f_j$. The limit is universal with respect to this property, in particular, for every other pair (N, g) satisfying the properties above, there exists a unique morphism $h : N \to L$, such that $f_i \circ h = g_i$, for every $i \in \mathcal{J}$. Dualise the above to obtain the definition of a small colimit of a diagram.

With this in mind we recall the following standard notions [41]:

- A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is called:

- *continuous* if it preserves small limits,
- *cocontinuous* if it preserves small colimits.

- A category $\mathcal{C}$ is called:
  - *(small-) complete* if all small diagrams have limits in $\mathcal{C}$,
  - *(small-) cocomplete* if all small diagrams have colimits in $\mathcal{C}$.

Theorem (The Freyd Adjoint Functor Theorem)**.** *[41, V.6, Theorem 2]*

Given a locally small, complete category $\mathcal{C}$ a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if it preserves all small limits and satisfies the following condition:

(SSC) For each object $d \in \mathcal{D}$ there is a set $\mathcal{I}$ and an $\mathcal{I}$-indexed family of morphisms $f_i : d \to \mathcal{F}(c_i)$, such that every morphism $h : d \to \mathcal{F}(c)$ can be written as a composite $h = \mathcal{F}(t) \circ f_i$, for some index $i \in \mathcal{I}$ and some $t : c_i \to c$.

Dualise the statement to obtain a criterion for a right adjoint.

We make use of the following:

Theorem 3.2.1. *[41, V.2, Theorem 1] If a category $\mathcal{C}$ has arbitrary products and all pairs of morphisms in $\mathcal{C}$ have equalizers, then $\mathcal{C}$ is complete.*

Dually, if $\mathcal{C}$ is closed under arbitrary coproducts and all pairs of morphisms have coequalizers, then $\mathcal{C}$ is cocomplete.

Thus, to check completeness (cocompleteness) of a category, it is enough to show that $\mathcal{C}$ has arbitrary products (coproducts), and all pairs of morphisms have equalizers (coequalizers).

In the sections to follow, we will use the results above to find criteria for existence of induced and coinduced representations of certain categories of continuous representations of a topological group G.

3.3 Frobenius reciprocity for discrete representations

Throughout the section G denotes a topological group, $H \leqslant G$ a closed subgroup and R an associative ring with identity.

Recall the category $\mathcal{M}_d(G)$ of discrete representations of G defined in Example 2.0.4. The objects of $\mathcal{M}_d(G)$ are pairs (π, V), where V is a topological R-module with respect to the discrete topology, and (π, V) is a continuous representation of G. Due to the R-modules being endowed with the discrete topology, the morphisms in $\mathcal{M}_d(G)$ are just R-module homomorphisms, intertwining with the G-action.

Since we know what the topology on V is, we can make the continuity condition in Definition 2.0.3 more explicit. Thus, we have the following:

Defintion 3.3.1. A *discrete representation* of G is a pair (π, V), such that V is a topological R-module with respect to the discrete topology, $\pi : V \to \mathrm{Aut}_R(V)$ is a homomorphism, and for every $v \in V$, $\phi^{-1}(v) = \{(\mathbf{g}, v) \mid \pi(\mathbf{g})v = v\}$ is open in G, where $\phi : G \times V \to V$ is given by $(\mathbf{g}, v) \mapsto \pi(\mathbf{g})v$. In other words, for every $v \in V$, there exists a non-empty open set $K_v \subset G$, such that $\pi(\mathbf{k})v = v$, for every $\mathbf{k} \in K_v$.

Note that 1_G always satisfies $\pi(1_G)v = v$. Since the topology on G is determined by the fundamental neighbourhoods of identity, without loss of generality assume that K_v is an open neighbourhood of 1_G. We could go even further - for every $v \in V$ we can construct an open subgroup $\widetilde{K_v} \leqslant G$ generated by the elements in K_v. Then clearly $\pi(\mathbf{k})v = v$, for every $\mathbf{k} \in \widetilde{K_v}$.

As before, for a closed subgroup $H \leqslant G$ we have a functor

$$\mathrm{Res}_H^G : \mathcal{M}_d(G) \to \mathcal{M}_d(H).$$

Now we are ready to start our investigation of induction and coinduction.

3.3.1 Coinduction in $\mathcal{M}_d(G)$

We start our investigation of adjoints to Res_H^G with the left adjoint, i.e., coinduction. We wish to use Freyd's Theorem. The first step is to show that arbitrary products exist in $\mathcal{M}_d(G)$. Take a collection $\{(\pi_i, V_i)\}_{i \in \mathcal{I}} \in \mathcal{M}_d(G)$, where $\mathcal{I}$ is an arbitrary set. Let $V \coloneqq \prod_{i \in \mathcal{I}} V_i$ denote the product of V_i, $i \in \mathcal{I}$, as R-modules. V is a discrete space with respect to the box topology. It also has an obvious G-module structure - G acts componentwise. More precisely, if we denote the elements of V by $(v_i)_{i \in \mathcal{I}}$, we have the following:

$$\mathbf{g} \cdot (v_i)_{i \in \mathcal{I}} = (\pi_i(\mathbf{g})v_i)_{i \in \mathcal{I}}, \text{ for all } \mathbf{g} \in G, v_i \in V_i.$$

Write $\pi : G \to \mathrm{Aut}_R(V)$, where $\pi(\mathbf{g})$ is given by $\pi(\mathbf{g}) : (v_i)_{i \in \mathcal{I}} \mapsto (\pi_i(\mathbf{g})v_i)_{i \in \mathcal{I}}$. This is a homomorphism. However, the map $\phi : G \times V \to V$ induced by π is not necessarily continuous:

Fix $v \in V$. Then $(v_i)_{i \in \mathcal{I}}$, for some $v_i \in V_i$. Since $V_i \in \mathcal{M}_d(G)$, for every $v_i \in V_i$, there exists an open neighbourhood K_{v_i} of 1_G, such that $\pi_i(\mathbf{k})v_i = v_i$, for all $\mathbf{k} \in K_{v_i}$ and $i \in \mathcal{I}$. Thus, $K_v = \bigcap_{i \in \mathcal{I}} K_{v_i}$ has the property that $\pi(\mathbf{k})v = v$, for all $\mathbf{k} \in K_v$. But as $\mathcal{I}$ was chosen arbitrarily K_v does not have to be open. Therefore, the representation is not continuous at v and $V \notin \mathcal{M}_d(G)$. However, we can consider an R-submodule $V^{sm} \leqslant V$, such that $V^{sm} \in \mathcal{M}_d(G)$. Let

$$V^{sm} \coloneqq \{v \in V \mid \exists \text{ non-empty } K_v \text{ open in } G, \text{ such that } \pi(\mathbf{k})v = v, \text{ for all } \mathbf{k} \in K_v\}.$$

This is called the *continuous part* of V. Note that by construction V^{sm} is a continuous representation of G.

Example 3.3.2. (cf. [4], [9]) It is common to take continuous parts of representations when we want to define the *smooth dual* of a continuous discrete representation of a locally compact totally disconnected group. For example, if $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $R = \mathbb{C}$ and $(\pi, V) \in \mathcal{M}_d(G)$, then the *contragredient representation* is defined as the continuous part of the representation (π^*, V^*), where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\pi^*(\mathbf{g})f : v \mapsto f(\pi(\mathbf{g}^{-1})v)$, for $v \in V, f \in V^*, \mathbf{g} \in G$.

We claim the following:

Lemma 3.3.3. *Every collection $\{(\pi_i, V_i)\}_{i \in \mathcal{I}} \in \mathcal{M}_d(G)$, with $\mathcal{I}$ an arbitrary set, has a product in $\mathcal{M}_d(G)$ given by $(\pi, (\prod\limits_{i \in \mathcal{I}} V_i)^{sm})$, where $\pi = (\pi_i)_{i \in \mathcal{I}}$ defined as above. In other words, $\mathcal{M}_d(G)$ is complete.*

Proof. Let $V^{sm} \coloneqq (\prod\limits_{i \in \mathcal{I}} V_i)^{sm}$. Denote by $p_i : V^{sm} \to V_i$ the canonical projections in $\mathcal{M}_d(G)$. Let $(\rho, A) \in \mathcal{M}_d(G)$ and let $f_i : A \to V_i$ be a family of morphisms in

$\mathcal{M}_d(G)$ indexed by $\mathcal{I}$. As V^{sm} is an R-submodule of $\prod_{i\in\mathcal{I}} V_i$ there exists a unique R-module homomorphism $f : A \to V^{sm}$, such that $p_i \circ f = f_i$, for all $i \in \mathcal{I}$. It is also a G-linear map:

$$p_i(f(\rho(\mathbf{g})a)) = f_i(\rho(\mathbf{g})a) = \pi_i(\mathbf{g})f_i(a) = \pi_i(\mathbf{g})(p_i(f(a))) = p_i(\pi(\mathbf{g})f(a)),\ \mathbf{g} \in G, a \in A,$$

i.e.,

$$f(\rho(\mathbf{g})a) = \pi(\mathbf{g})f(a).$$

Thus, f is a morphism in $\mathcal{M}_d(G)$ and the universal property of the product is satisfied. □

We claim that even though $\mathcal{M}_d(G)$ is complete, the restriction functor does not always have a left adjoint.

Theorem 3.3.4. *Let G be a topological group and H a closed subgroup of G. Then*

$$\mathrm{Res}_H^G : \mathcal{M}_d(G) \to \mathcal{M}_d(H)$$

has a left adjoint if H is also open.

Proof. We wish to use Freyd's Theorem to establish the result. First note that (SSC) holds in $\mathcal{M}_d(G)$: it just says that every map in $\mathcal{M}_d(G)$ can be factored through a quotient. Also since $\mathcal{M}_d(G)$ is abelian (cf. [9]), equalizers of all morphisms exist. By Lemma 3.3.3 arbitrary products also exist. Moreover, since Res_H^G does not change the morphisms between objects, it commutes with equalizers. Thus, to establish continuity of Res_H^G we need to determine when it commutes with arbitrary products, i.e., for a collection $\{(\pi_i, V_i)\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is an arbitrary set, when does the following hold:

$$\left(\prod_{i\in\mathcal{I}} \mathrm{Res}_H^G(V_i)\right)^{sm} \cong \mathrm{Res}_H^G\left((\prod_{i\in\mathcal{I}} V_i)^{sm}\right).$$

This is the same as showing that every $(\sigma, W) \in \mathcal{M}_d(H)$, such that $W = \left(\prod_{i\in\mathcal{I}} \mathrm{Res}_H^G((\pi_i, V_i))\right)^{sm}$, $\sigma = (\pi_i|_H)_{i\in\mathcal{I}}$, for some $(\pi_i, V_i) \in \mathcal{M}_d(G)$, is also an element of $\mathcal{M}_d(G)$.

Take such $(\sigma, W) \in \mathcal{M}_d(H)$. By definition we can write $w = (v_i)_{i\in\mathcal{I}}$, for some $v_i \in \mathrm{Res}_H^G(V_i)$. Note that as an R-module $\mathrm{Res}_H^G(V_i) = V_i$ and G acts on each V_i via π_i. So to show that $(W, \sigma) \in \mathcal{M}_d(G)$, we need to establish that the representation is continuous. Thus, we need to study the continuity condition. Since $(\sigma, W) \in \mathcal{M}_d(H)$, for every $w \in W$ there exists an open neighbourhood K_w of 1_H, such that $\sigma(\mathbf{k})w = w$, for all $\mathbf{k} \in K_w$. We need to show that K_w is open in G. As H is given the subspace topology, there is an open $U \subseteq G$, such that $K_w = U \cap H$. But since H is open in G, then K_w is an open neighbourhood of 1_G and thus $(\sigma, W) \in \mathcal{M}_d(G)$. $\square$

Thus, for an open subgroup $H \leqslant G$, we have a functor

$$\mathrm{Coind}_H^G : \mathcal{M}_d(H) \to \mathcal{M}_d(G),$$

which is left adjoint to the restriction functor.

Example 3.3.5. Suppose the topology on G is locally compact and totally disconnected and $H \leqslant G$ is an open subgroup. Then Coind_H^G is given by *compact induction of representations*, usually denoted $c - \mathrm{Ind}_H^G$, i.e.,

$$\mathrm{Coind}_H^G = c - \mathrm{Ind}_H^G : \mathcal{M}_d(H) \to \mathcal{M}_d(G), \quad (\sigma, W) \mapsto (\rho, \widetilde{W}),$$

where $\widetilde{W}$ is the space of all left H-equivariant, continuous functions $f : G \to W$, which have compact modulo H support, and $\rho(\mathbf{g})f : \mathbf{x} \to f(\mathbf{xg})$ [9, 1.3.4].

The next example shows that if H is not open in G, then Coind_H^G is not always defined.

Example 3.3.6. Take $\{(\pi_i, V_i)\}_{i\in\mathcal{I}} \in \mathcal{M}_d(G)$. Fix $v = (v_i)_{i\in\mathcal{I}} \in \prod\limits_{i\in\mathcal{I}} V_i$ with K_i corresponding open neighbourhoods of 1_G, such that $\pi_i(\mathbf{k}_i)v_i = v_i$, for all $\mathbf{k}_i \in K_i$. For each $i \in \mathcal{I}$ construct an open subgroup $\widetilde{K_i} \leqslant G$, generated by K_i. Let $\pi = (\pi_i)_{i\in\mathcal{I}}$. Then $\pi(\mathbf{k})v = v$, for all $\mathbf{k} \in \widetilde{K} := \bigcap_{i\in\mathcal{I}} \widetilde{K_i}$. However, as $\mathcal{I}$ is an arbitrary set $\widetilde{K}$ is not necessarily open in G. Moreover, as every open subgroup of a topological group is closed [30, 5.5], it follows that each $\widetilde{K_i}$ is closed, and thus so is $\widetilde{K}$. Taking $H \coloneqq \widetilde{K}$ we have $v \notin (\prod_{i\in\mathcal{I}} V_i)^{sm}$, and thus $v \notin \mathrm{Res}^G_H(\prod_{i\in\mathcal{I}} V_i)^{sm}$. But as H is open in H, $v \in \big(\prod_{i\in\mathcal{I}}(\mathrm{Res}^G_H(V_i)\big)^{sm}$. Thus, Res^G_H fails to be continuous and Coind^G_H is not defined.

3.3.2 Induction in $\mathcal{M}_d(G)$

To define induction, we follow a different method. For every representation $(\sigma, W) \in \mathcal{M}_d(H)$, we construct the induced representation $\mathrm{Ind}^G_H((\sigma, W))$ explicitly and then show that the functor defined by this procedure is right adjoint to Res^G_H. Again throughout the section G is a topological group, $H \leqslant G$ a closed subgroup, and R an associative ring with identity.

Consider the following construction, which is a generalisation of the construction for a smoothly induced representation for locally compact totally disconnected topological groups [9], [32]:

Fix $(\sigma, W) \in \mathcal{M}_d(H)$. Consider the R-module $\widehat{W}$ of all left H-equivariant functions $f : G \to W$, i.e., which satisfy the property

(i) $f(\mathbf{hg}) = \sigma(\mathbf{h})f(\mathbf{g})$, for all $\mathbf{h} \in H$ and $\mathbf{g} \in G$.

Within $\widehat{W}$ we find an R-submodule $\widetilde{W}$ consisting of "continuous functions", i.e., functions with the additional property

(ii) $f \in \widetilde{W}$ if and only if there exists an open neighbourhood K_f of 1_G, such that $f(\mathbf{gk}) = f(\mathbf{g})$, for all $\mathbf{g} \in G$ and $\mathbf{k} \in K_f$.

The homomorphism $\rho : G \to \mathrm{Aut}_R(\widehat{W})$, given by $\rho(\mathbf{g})f : \mathbf{x} \mapsto f(\mathbf{xg})$, for $\mathbf{g}, \mathbf{x} \in G$ and $f \in \widehat{W}$, defines a G-action on both $\widehat{W}$ and $\widetilde{W}$, thus making $(\rho, \widetilde{W})$ a continuous representation of G, i.e., $(\rho, \widetilde{W}) \in \mathcal{M}_d(G)$. The pair $(\rho, \widetilde{W})$ is called the representation of G *continuously induced by* σ. Using this construction we define the functor $\mathrm{Ind}^G_H : \mathcal{M}_d(H) \to \mathcal{M}_d(G)$. We claim that this is the right adjoint we are looking for.

Lemma 3.3.7. *For a topological group G and a closed subgroup $H \leqslant G$, the functor $\mathrm{Ind}^G_H : \mathcal{M}_d(H) \to \mathcal{M}_d(G)$ defined above is right adjoint to the restriction functor.*

Proof. For continuous representations $(\pi, V) \in \mathcal{M}_d(G)$ and $(\sigma, W) \in \mathcal{M}_d(H)$ and notation as above, we want

$$\mathrm{Hom}_{\mathcal{M}_d(G)}((\pi, V), (\rho, \widetilde{W})) \cong Hom_{\mathcal{M}_d(H)}((\pi|_H, {}_HV)),$$

where ${}_HV$ denotes V as an H-module. We have maps:

$$\alpha : \mathrm{Hom}_{\mathcal{M}_d(G)}((\pi, V), (\rho, \widetilde{W})) \to \mathrm{Hom}_{\mathcal{M}_d(H)}((\pi|_H, {}_HV), (\sigma, W)),$$

given by

$$(\psi : V \to \widetilde{W}) \quad \mapsto \quad (\widetilde{\psi} : V \to W),$$

where

$$(\psi : v \mapsto \psi_v) \quad \mapsto \quad (\widetilde{\psi} : v \mapsto \psi_v(1_G)),$$

and $\beta : \mathrm{Hom}_{\mathcal{M}_d(H)}((\pi|_H, {}_HV), (\sigma, W)) \to \mathrm{Hom}_{\mathcal{M}_d(G)}((\pi, V), (\rho, \widetilde{W}))$ given by:

$$(\phi : V \to W) \quad \mapsto \quad (\widetilde{\phi} : V \to \widetilde{W})$$

with

$$\widetilde{\phi} : v \mapsto (f_v : \mathbf{g} \mapsto \phi(\pi(\mathbf{g})v)).$$

Clearly $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{M}_d(H)}((\pi|_H, {}_HV), (\sigma, W))$ since $\psi_v \in \widetilde{W}$.
Similarly, $\widetilde{\phi} \in \mathrm{Hom}_{\mathcal{M}_d(G)}((\pi, V), (\rho, \widetilde{W}))$. It is routine to check that α and β are inverse to each other. $\square$

Note that it follows directly from the constructions that for a locally compact totally disconnected group G and $H \leqslant G$, such that $H\backslash G$ is compact open, $\mathrm{Ind}_H^G \cong \mathrm{Coind}_H^G$ (cf. Example 3.3.5, [9], [55]).

3.4 Frobenius reciprocity for linearly topologized complete representations

Again fix an associative unital ring R, a topological group G and a closed subgroup $H \leqslant G$. We are interested in investigating induction and coinduction in the category of linearly topologized and complete continuous representations $\mathcal{M}_{ltc}(G)$, defined in Example 2.0.6. Again, since our topological R-modules are endowed with a specific topology, we can rephrase the continuity condition of Definition 2.0.3:

Defintion 3.4.1. A *linearly topologized and complete representation* of G is a pair (π, V), such that $(V, \mathcal{T}_V)$ is a topological R-module with $\mathcal{T}_V$ a linear and complete topology, $\pi : V \to \mathrm{Aut}_R(V)$ is a homomorphism, and for every open R-submodule $U \leqslant V$, such that there exists $\mathbf{g} \in G$ and $x \in V$ with $\pi(\mathbf{g})x \in U$, there is an open neighbourhood K of 1_G and an open R-submodule $W \leqslant V$ satisfying $\pi(K\mathbf{g})(x + W) \subseteq U$.

Having made the definition of continuous representation in $\mathcal{M}_{ltc}(G)$ precise, we move on to investigate completeness and cocompleteness of $\mathcal{M}_{ltc}(G)$. We will follow our strategy from Section 3.3.1 - we establish completeness and cocompleteness by establishing existence of arbitrary products and coproducts in $\mathcal{M}_{ltc}(G)$ and then we apply Freyd's Theorem.

3.4.1 Products and Coproducts in $\mathcal{M}_{ltc}(G)$

In this section we keep the conventions as above - R is an associative unital ring, G a topological group, $H \leqslant G$ a closed subgroup. We start by looking at arbitrary products in $\mathcal{M}_{ltc}(G)$. Our first result of the section shows their existence.

Lemma 3.4.2. *Arbitrary products exist in $\mathcal{M}_{ltc}(G)$. More precisely, the product of a collection of objects of $\mathcal{M}_{ltc}(G)$ is their product as R-modules, endowed with the product topology.*

Proof. For an arbitrary collection $\{(\pi_i, V_i)\}_{i \in \mathcal{I}}$ of objects of $\mathcal{M}_{ltc}(G)$, let $V \coloneqq \prod_{i \in \mathcal{I}} V_i$ denote their product as R-modules, which is a topological R-module with respect to the product topology [8, III.6.6].

Following Lefschetz we show that the topology on V is linear [40, 25.3]. Let $\{U_i^j\}, j \in \mathcal{J}_i$ be a base of neighbourhoods of 0 in V_i, consisting of open submodules. Then for any finite subset $\mathcal{K} \subset \mathcal{I}$, $\prod_{k\in\mathcal{K}} U_k^j \times \prod_{i\in\mathcal{I}\setminus\mathcal{K}} V_i$ is a base of neighbourhoods of 0 in V consisting of open submodules, giving the linearity of the product topology.

Let $(z_n) \coloneqq ((z_n^i))_{i\in\mathcal{I},n\in\mathbb{L}}$, where (z_n^i) are nets in V_i and $\mathbb{L}$ is an ordinal. Suppose (z_n) is a Cauchy net in V. Thus, every net $(z_n^i)_{n\in\mathbb{L}}$ in V_i is Cauchy. Since all V_i are complete, each $(z_n^i)_{n\in\mathbb{L}} \in V_i$ is a convergent net in V_i. Let

$$z^i \coloneqq \lim_{n\in\mathbb{L}} z_n^i.$$

Set $z \coloneqq (z^i)_{i\in\mathcal{I}}$. Let U_i be an open neighbourhood of z^i in V_i and define $U = \prod_{i\in\mathcal{J}} U_i \times \prod_{i\in I\setminus\mathcal{J}} V_i$, for some finite subset $\mathcal{J} \subset \mathcal{I}$. Since each net $(z_n^i)_{n\in\mathbb{L}}$ is convergent, there exists some l_i, such that $z_n^i \in U_i$, for all $n \geqslant l_i$ in $\mathbb{L}$. Pick the largest $l_i, i \in \mathcal{J}$, say l. Then for all $n \geqslant l$ in $\mathbb{L}$, $z_n^i \in U_i$ for all $i \in \mathcal{I}$. Thus, $z \in U$ and (z_n) is convergent in V with

$$\lim_{n\in\mathbb{L}} z_n = z.$$

The G-action on V is componentwise, i.e., $\mathbf{g} \cdot v \coloneqq \pi(\mathbf{g})v$, for $\mathbf{g} \in G$ and $v = (v_i)_{i\in\mathcal{I}} \in V$, where $\pi = (\pi_i)_{i\in\mathcal{I}}$. We want to show it is continuous. Let $U \coloneqq \prod_{i\in\mathcal{J}} U_i \times \prod_{i\in\mathcal{I}\setminus\mathcal{J}} V_i$ with $\mathcal{J} \subset \mathcal{I}$ finite. Then $U \subseteq V$ is open. Since all V_i are continuous G-modules, for every $i \in \mathcal{I}$ there exists an open neighbourhood N_i of 1_G and an open submodule $W_i \leqslant V_i$, such that if $\pi_i(\mathbf{g})x_i \in U_i$, for $\mathbf{g} \in G, x_i \in V_i$, then $\pi_i(N_i\mathbf{g})(x_i + W_i) \subseteq U_i$. Fix $N \coloneqq \bigcap_{i\in\mathcal{J}} N_i$. Since $\mathcal{J}$ is finite, N is an open neighbourhood of 1_G and furthermore $\pi_i(N\mathbf{g})(x_i + W_i) \subseteq U_i$, for all $i \in \mathcal{J}$. Let $W \coloneqq \prod_{i\in\mathcal{J}} W_i \times \prod_{i\in\mathcal{I}\setminus\mathcal{J}} V_i$. This is an open submodule of V. Thus, we found $N \subseteq G$ and an open submodule $W \leqslant V$, such that for $\pi(\mathbf{g})x \in U$, with $x = (x_i)_{i\in\mathcal{I}} \in V$, $\pi(N\mathbf{g})(x + W) \in U$. Hence, $(\pi, V) \in \mathcal{M}_{ltc}(G)$.

Let $(\rho, A) \in \mathcal{M}_{ltc}(G)$, $p_i : V \to V_i$ be the projections in $\mathcal{M}_{ltc}(G)$ and $f_i : A \to V_i$ be a family of morphisms in $\mathcal{M}_{ltc}(G)$ indexed by $\mathcal{I}$. As V is the product of $\{V_i\}_{i\in\mathcal{I}}$ as R-modules, there exists a unique R-module homomorphism $f : A \to V$, making the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "p_i"] & V_i \\ & A \arrow[u, "f_i"'] \arrow[ul, dashed, "f"] \end{tikzcd}$$

The map f has the following properties:

1. $(p_i \circ f)(\rho(\mathbf{g})a) = f_i(\rho(\mathbf{g})a) = \pi_i(\mathbf{g})f_i(a) = \pi_i(\mathbf{g})((p_i \circ f)(a))$, for $\mathbf{g} \in G$ and $a \in A$, i.e., f is G-linear.

2. For $U \subseteq V$ open, $U_i \coloneqq p_i(U) \subseteq V_i$ is open. By continuity of f_i it follows that $f_i^{-1}(U_i) \subseteq A$ is open, for every $i \in \mathcal{I}$. Thus, $f^{-1}(U) = f_i^{-1}(p_i(U)) \subseteq A$ is open, showing that f is continuous.

Thus, f is a morphism in $\mathcal{M}_{ltc}(G)$, finishing the proof. □

To continue our investigation of adjoint functors, we would also need existence of arbitrary coproducts in $\mathcal{M}_{ltc}(G)$. We construct them explicitly. Let $\{(\pi_i, V_i)\}_{i\in\mathcal{I}}$ be an arbitrary collection of objects of $\mathcal{M}_{ltc}(G)$. Denote by $V \coloneqq \bigoplus_{i\in\mathcal{I}} V_i$ their coproduct as R-modules and by $\alpha_i : V_i \to V$ the canonical injections. In this case they are just inclusion maps. We follow Higgins in defining the topology on V [31]. Consider pairs (W, τ_W), such that:

1. $W \in \mathcal{M}_{ltc}(G)$, such that there exists a surjective R-module homomorphism $q_W : V \to W$ which is also G-linear,

2. τ_W is a topology on W in which the maps $q_W^i : V_i \to W$ that factor through q_W are continuous.

All such pairs (W, τ_W) taken up to isomorphism form a set. Hence, we can form a product $\prod_{(W,\tau_W)} W$. The map

$$q : V \to \prod_{(W,\tau_W)} W, \text{ given by } v \mapsto (q_W(v))_{(W,\tau)} \tag{3.1}$$

is an embedding. We endow $\prod_{(W,\tau_W)} W$ with the product topology and V with the topology induced by q. This topology makes V into a topological abelian group

[31]. The map $\phi : R \times \prod_{(W,\tau_W)} W \to \prod_{(W,\tau_W)} W$ is continuous, hence, the restriction $\phi\mid_{q(V)}: R \times q(V) \to q(V)$ is also continuous. Thus, the subspace topology on $q(V)$, and respectively the induced one on V, makes V into a topological R-module. By Lemma 3.4.2 $\prod_{(W,\tau_W)} W$ lies in $\mathcal{M}_{ltc}(G)$. Every subspace of a linearly topologized space is linearly topologized [40, 25.2]. Thus, as $V \cong q(V)$, both as an R-module and as a topological space, the topology on it is linear. A priori V is not necessarily complete. However, its closure $\overline{V}$ is, as it is a closed subspace of a complete space [8, II.4.8].

Lemma 3.4.3. *$(\pi, \overline{V})$, where $\overline{V}$ is as above and $\pi = (\pi_i)_{i\in\mathcal{I}}$, is the coproduct of $\{(\pi_i, V_i)\}_{i\in\mathcal{I}}$ in $\mathcal{M}_{ltc}(G)$.*

Proof. By definition $\overline{V}$ is a linearly topologized and complete space. As $V = \bigoplus_{i\in\mathcal{I}} V_i$ and V_i is a G-module for every $i \in \mathcal{I}$, then clearly so is V with respect to the componentwise action of G. Since $\prod_{(W,\tau_W)} W \in \mathcal{M}_{ltc}(G)$, the map $G \times \prod_{(W,\tau_W)} W \to \prod_{(W,\tau_W)} W$ is continuous. Hence, its restriction to a subspace is also continuous. Therefore, V is a continuous G-module and hence, so is its closure $\overline{V}$. Thus, $(\pi, \overline{V}) \in \mathcal{M}_{ltc}(G)$ as required.

Let us check that $(\pi, \overline{V})$ is indeed the coproduct of $\{(\pi_i, V_i)\}_{i\in\mathcal{I}}$. Let (ρ, A) be any module in $\mathcal{M}_{ltc}(G)$ and $\beta_i : V_i \to A$ be morphisms in $\mathcal{M}_{ltc}(G)$ indexed by $\mathcal{I}$. Since V is the coproduct of $\{V_i\}_{i\in\mathcal{I}}$ as R-modules, there exists a unique R-module homomorphism $f : \overline{V} \to A$, such that for every $i \in \mathcal{I}$ the diagram below commutes:

$$\begin{tikzcd} V_i \arrow[r, "\alpha_i"] \arrow[dr, "\beta_i"'] & \overline{V} \arrow[d, dashed, "f"] \\ & A \end{tikzcd}$$

The map f is also G-linear:

$$(f \circ \alpha_i)(\pi_i(\mathbf{g})v_i) = f(\pi(\mathbf{g})(\alpha_i(v_i))) = \beta_i(\pi_i(\mathbf{g})v_i) = \rho(\mathbf{g})\beta_i(v_i) = \rho(\mathbf{g})f(\alpha_i(v_i)),$$

for $v_i \in V_i$, $\mathbf{g} \in G$.

Lastly, let $U \subseteq A$ be open. By continuity of β_i, $\beta_i^{-1}(U) \subseteq V_i$ is open, for every $i \in \mathcal{I}$. Since the V_i's appear amongst the (W, τ_W), then $\beta_i^{-1}(U) \times \prod_{(W,\tau_W), W\neq V_i} W$ is open in $\prod_{(W,\tau_W)} W$ and moreover $\beta_i^{-1}(U) \times \prod_{\substack{(W,\tau_W),\\ W\neq V_i}} W = q^i(\beta_i^{-1}(U))$. Thus,

$$q^{-1}\big(\beta_i^{-1}(U) \times \prod_{\substack{(W,\tau_W),\\ W\neq V_i}} W\big) = q^{-1}(q^i(\beta_i^{-1}(U))) = \alpha_i(\beta_i^{-1}(U)) = f^{-1}(U),$$

where q^i is given componentwise by the q^i_W defined above. By definition of the topology on $\overline{V}$ it follows that f is continuous, finishing the proof. $\square$

Recall that for a set X, topological spaces X_i and maps $f_i : X_i \to X$, the *final topology* on X is the finest topology on X in which all f_i are continuous [18]. With notation as before, we have the diagram:

$$\begin{tikzcd} V_i \arrow[r, "\alpha_i"] \arrow[rr, bend right, "q^i"'] & V \arrow[r, "q"] & \prod\limits_{(W,\tau_W)} W. \end{tikzcd}$$

Since q^i_W is continuous for every $i \in \mathcal{I}$, then so is q^i [8]. By definition of the topology on V, q is continuous. Hence, α_i is continuous for each i. This means that the topology on V is contained in the final topology with respect to α_i. However, the continuity of the α_i implies that V appears as one of the W, thus, the coproduct topology defined above coincides with the final topology.

Now we would like to give an explicit description of the basis of open neighbourhoods of 0 in V. Chasco and Domínguez describe this basis with respect to the final topology for a coproduct of topological abelian groups [18]. We generalise their construction to topological R-modules.

Let $\{U_i\}_{i\in\mathcal{I}}$ be a sequence of neighbourhoods of 0, with U_i a neighbourhood of 0 in V_i. Let $\mathcal{J} \subseteq \mathcal{I}$ be finite. Then

$$U \coloneqq \bigcup_{\substack{n\in\mathbb{N},\\ |\mathcal{J}|=n}} \bigcup_{\substack{\mathcal{K}\subseteq\mathcal{I},\\ |\mathcal{K}|=|\mathcal{J}|}} \sum_{i\in\mathcal{K}} \alpha_i(U_i) \tag{3.2}$$

is a sequence of neighbourhoods of 0 in V. Hence, the basis is given by

$$\mathcal{U} = \{U \mid \{U_i\}_{i\in\mathcal{K}}, \text{ with } U_i \subseteq V_i \text{ open neighbourhood of } 0\}.$$

This indeed agrees with our description of the topology: If $\mathcal{B} \in \prod_{(W,\tau_W)} W$ is an open neighbourhood of 0 in $\prod_{(W,\tau_W)} W$, $q^{-1}(\mathcal{B})$ would be the corresponding open in V and

$$q^{-1}(\mathcal{B}) \bigcap V = q^{-1}(\mathcal{B}) \bigcap \bigoplus_{i\in\mathcal{I}} V_i = \bigoplus_{i\in\mathcal{I}} q^{-1}(\mathcal{B}) \bigcap V_i,$$

can be written in the form of (3.2).

3.4.2 Existence of induction and coinduction

Before proceeding to the main result of the section, we make a final observation about $\mathcal{M}_{ltc}(G)$.

Lemma 3.4.4. *The category $\mathcal{M}_{ltc}(G)$ is additive. Moreover, all morphisms have kernels and cokernels.*

Proof. The category $\mathcal{M}_{ltc}(G)$ is clearly pre-additive, has a zero object and existence of arbitrary, hence finite, products and coproducts is guaranteed by Lemma 3.4.2 and Lemma 3.4.3. Thus, $\mathcal{M}_{ltc}(G)$ is additive.

Let $f : V \to W$ be a morphism in $\mathcal{M}_{ltc}(G)$. Let K be the kernel of f as an R-module map. Clearly, K is an R-submodule of V. It is also a G-submodule as f is G-linear. The restriction of the map $\phi : G \times V \to V$, given by $(\mathbf{g}, v) \mapsto \mathbf{g} \cdot v$, to $G \times K$ is continuous, since ϕ is continuous. Since every submodule of a linearly topologized module is linearly topologized ([40, 25.2]), K is linearly topologized. By definition $K = f^{-1}(\{0_W\})$. Since W is Hausdorff, $\{0_W\}$ is closed and thus K is closed. A closed subspace of a complete space is complete and so $K = \ker(f) \in \mathcal{M}_{ltc}(G)$.

Let $I = \mathrm{im}(f)$ and $C = \mathrm{coker}(f) = W/I$ as R-modules. I and C are topological R-modules with respect to the subspace topology. Both I and C are clearly G-modules. The G-action is continuous, as $\phi|_I : G \times I \to I$ is a restriction of ϕ, hence continuous. The map $\phi_{W/I} : G \times W/I \to W/I$ is continuous since ϕ is continuous and W/I is endowed with the quotient topology induced by the topology on W. A submodule and a quotient of linearly topologized modules are linearly topologized ([40, 25.2, 25.3]), so we only have to show that C is complete.

This is equivalent to showing that I is closed in W. Let $(x_n)_{n \in \mathbb{L}}$ be a Cauchy net in V, $\mathbb{L}$ is an ordinal. By completeness, $(x_n)_{n \in \mathbb{L}}$ is convergent. Let $x = \lim_{n \in \mathbb{L}} x_n$. By continuity of f the net $(f(x_n))_{n \in \mathbb{L}}$ is convergent in W. A limit of it is $f(x)$. However, as W is Hausdorff, limits are unique and so $f(x) \in I$ is the only limit, showing that $I \leqslant W$ is closed. Thus, $C = W/I$ is complete and so $C \in \mathcal{M}_{ltc}(G)$, finishing the proof. $\square$

We are ready to address the main problem of this section.

Theorem 3.4.5. *Let G be a topological group and $H \leqslant G$ a closed subgroup. Let*

$$\mathrm{Res}_H^G : \mathcal{M}_{ltc}(G) \to \mathcal{M}_{ltc}(H)$$

be the restriction functor. The following hold:

1. Res_H^G *has a left adjoint* $\mathrm{Coind}_H^G : \mathcal{M}_{ltc}(H) \to \mathcal{M}_{ltc}(G)$.
2. *If H is also open, then* Res_H^G *has a right adjoint*

   $$\mathrm{Ind}_H^G : \mathcal{M}_{ltc}(H) \to \mathcal{M}_{ltc}(G),$$

   given by $\mathrm{Ind}_H^G : W \mapsto \mathrm{Fun}_H(G, W)$, *where* $\mathrm{Fun}_H(G, W)$ *is the space of all functions* $f : G \to W$, *such that* $f(\mathbf{gh}) = \mathbf{h} \cdot f(\mathbf{g})$.

Proof. We wish to apply Freyd's Adjoint Functor Theorem to prove the statements. Since $\mathcal{M}_{ltc}(G)$ has kernels and cokernels of all morphisms (Lemma 3.4.4), it has equalizers and coequalizers of all pairs of morphisms [41, VII.3]. By the product-equalizer (coproduct-coequalizer) construction of limits (respectively colimits) (Theorem 3.2.1), Lemma 3.4.2 and Lemma 3.4.3 imply that $\mathcal{M}_{ltc}(G)$ is both complete and cocomplete. Since the restriction functor does not change morphisms, it commutes with equalizers and coequalizers. Thus, to show existence of a left and a right adjoint to Res_H^G we only need to check whether it commutes with products and coproducts. Let us start with products, i.e., we want

$$\operatorname{Res}_H^G \big(\prod_{i\in\mathcal{I}}(\pi_i, V_i)\big) \cong \prod_{i\in\mathcal{I}} \operatorname{Res}_H^G((\pi_i, V_i)),$$

where $(\pi_i, V_i) \in \mathcal{M}_{ltc}(G)$, for all $i \in \mathcal{I}$, and $\mathcal{I}$ is an arbitrary set.

By Lemma 3.4.2 $\prod_{i\in\mathcal{I}} \operatorname{Res}_H^G((\pi_i, V_i)) \in \mathcal{M}_{ltc}(H)$ and looking at the R-modules we see that

$$\prod_{i\in\mathcal{I}} \operatorname{Res}_H^G(V_i) \cong \prod_{i\in\mathcal{I}} {}_H V_i \cong {}_H\big(\prod_{i\in\mathcal{I}} V_i\big) \cong \operatorname{Res}_H^G \big(\prod_{i\in\mathcal{I}} V_i\big).$$

Note that ${}_H(\prod_{i\in\mathcal{I}} V_i)$ is already a continuous H-module. Thus, the second isomorphism holds because $\prod_{i\in\mathcal{I}} {}_H V_i$ and ${}_H(\prod_{i\in\mathcal{I}} V_i)$ have the same structure as H-modules, as well as topological spaces. The H-actions on both sides are the same, as they are restrictions of the same G-actions π_i. This completes the proof of (1).

Now we move on to coproducts. To have a right adjoint to Res_H^G we need to check cocontinuity, i.e.:

$$\bigoplus_{i\in\mathcal{I}} \operatorname{Res}_H^G(V_i) \cong \operatorname{Res}_H^G \big(\bigoplus_{i\in\mathcal{I}} V_i\big),$$

where by $\oplus$ we denote the coproduct in $\mathcal{M}_{ltc}(H)$ and $\mathcal{M}_{ltc}(G)$ respectively. This amounts to showing that

$$\bigoplus_{i\in\mathcal{I}} {}_H V_i \cong {}_H\big(\bigoplus_{i\in\mathcal{I}} V_i\big).$$

Since $\bigoplus_{i\in\mathcal{I}} {}_H V_i$ and ${}_H \bigoplus_{i\in\mathcal{I}} V_i$ have the same structure as H-modules and the action of H on both is continuous, to obtain the isomorphism, we need to show that the coproduct topologies on both sides are the same.

In particular, for every H-module (W, τ_W) for which there exists a surjective H-map $\varphi : \bigoplus_{i\in\mathcal{I}} {}_H V_i \to W$ with the property that the maps $q_i : V_i \to W$ are continuous, we have to find a module $(\widetilde{W}, \tau_{\widetilde{W}}) \in \mathcal{M}_{ltc}(G)$, for which there exists a surjective G-map $\widetilde{\varphi} : {}_H \bigoplus_{i\in\mathcal{I}} V_i \to \widetilde{W}$, such that the maps $\widetilde{q}_i : V_i \to \widetilde{W}$ factoring through $\widetilde{\varphi}$ are continuous. In addition, for every open $U \subseteq W$ there must be an open $\widetilde{U} \subseteq \widetilde{W}$, such that $\widetilde{\varphi}^{-1}(\widetilde{U}) \subseteq \varphi(U)$.

Fix (W, τ_W) with the above properties. Let $\mathrm{Fun}_H(G, W)$ be the space of all right H-equivariant functions $f : G \to W$, i.e., which satisfy the property:

$$f(\mathbf{gh}) = \mathbf{h} \cdot f(\mathbf{g}), \text{ for } \mathbf{g} \in G, \mathbf{h} \in H.$$

This is a G-module via $\mathbf{g} \cdot f : \mathbf{x} \mapsto f(\mathbf{xg})$. We claim that $\widetilde{W} := \mathrm{Fun}_H(G, W)$ satisfies the desired conditions. First, we show that $\widetilde{W} \in \mathcal{M}_{ltc}(G)$. Let X be a set of right coset representatives of H in G. There is an H-module isomorphism:

$$\psi : \mathrm{Fun}_H(G, W) \cong \prod_{a \in X} a \otimes_H W, \quad \psi : f \mapsto (a_i \otimes f(1))_{a_i \in X}.$$

Identifying the right hand-side as $|X|$ copies of W, we can put the product topology on it. By Lemma 3.4.2 $\prod_{a \in X} a \otimes_H W \in \mathcal{M}_{ltc}(H)$. Endow $\mathrm{Fun}_H(G, W)$ with the topology induced by ψ. Since $\mathrm{Fun}_H(G, W) \cong \prod_{a \in X} a \otimes_H W$ as H-modules and they have the same topology, then $\mathrm{Fun}_H(G, W) \in \mathcal{M}_{ltc}(H)$.

Let $U \subseteq \mathrm{Fun}_H(G, W)$ be open. Suppose $\mathbf{g} \cdot f \in U$, for $\mathbf{g} \in G$ and $f \in \mathrm{Fun}_H(G, W)$. The function $\mathbf{g} \cdot f$ is H-equivariant and we can rewrite it as $\mathbf{g} \cdot f : \mathbf{x} \mapsto f(\mathbf{xg}) = f(\mathbf{yh}) = \mathbf{h} \cdot f(\mathbf{y}) := \mathbf{h} \cdot w_{\mathbf{y}} \in U$, for $\mathbf{g}, \mathbf{x}, \mathbf{y} \in G$, $\mathbf{h} \in H$ and $w_{\mathbf{y}} \in W$. Since W is a continuous representation of H, there exists an open neighbourhood K of 1_H and an open submodule $Z \leqslant W$, such that $K\mathbf{h}(w_{\mathbf{y}} + Z) \in U$. But as $H \leqslant G$ is open, K is an open neighbourhood of 1_G. Set $\widetilde{Z} := \psi^{-1}(1_G \otimes Z \times \prod_{a \in X, a \neq 1_G} a \otimes W)$. This is an open submodule of $\mathrm{Fun}_H(G, W)$. Moreover, the pair $(K, \widetilde{Z})$ has the property $K\mathbf{g}(f + \widetilde{Z}) \subseteq U$. In particular, $\mathrm{Fun}_H(G, W)$ is a continuous representation of G.

Next, extend every surjective H-map $\varphi : \bigoplus_{i \in \mathcal{I}} {}_H V_i \to W$ to a surjective G-map $\widetilde{\varphi} : {}_H \bigoplus_{i \in \mathcal{I}} V_i \to \widetilde{W}$ by

$$\widetilde{\varphi} : v \mapsto (f_v : \mathbf{g} \to \varphi(\mathbf{g} \cdot v)).$$

Fix an open $U \subseteq W$. Then $b \otimes U \times \prod_{a \in X, a \neq b} a \otimes W \subseteq \prod_{a \in X} a \otimes W$ is open. Let $\widetilde{U} = \psi^{-1}(b \otimes U \times \prod_{a \in X, a \neq b} a \otimes W)$. By definition $\widetilde{U}$ is open in $\widetilde{W}$ and $\widetilde{U} = \{f \in \mathrm{Fun}_H(G, W) \mid f(1) \in U\}$. Then

$\widetilde{\varphi}^{-1}(\widetilde{U}) = \{v \in V \mid f_v \in \widetilde{U}\} = \{v \in V \mid f_v(1) \in U\} \subseteq \{v \in V \mid \varphi(v) \in U\} = \varphi^{-1}(U)$.

□

3.5 Frobenius reciprocity for compact representations

In this section we tackle Frobenius reciprocity in the category $\mathcal{M}_c(G)$ defined in Example 2.0.7. We split our investigation into two cases - in the first case our associative ring R is a field, and in the second we consider a general ring R which is associative and with unity. As always, throughout the section G denotes a topological group and $H \leqslant G$ is a closed subgroup.

3.5.1 Induction and coinduction when $R = \mathbb{F}$

Suppose $R = \mathbb{F}$, where $\mathbb{F}$ is a field and let $(V, \mathcal{T}_V)$ be a topological vector space over $\mathbb{F}$, where the topology $\mathcal{T}_V$ is (linearly) compact (cf. Example 2.0.7). The notion of linear compactness of vector spaces first appears in Lefschetz' "Algebraic Topology" [40]. He calls a vector space linearly compact if it is linearly topologized, Hausdorff, and satisfies the finite intersection property on cosets of closed subspaces. Such spaces are complete [40]. This leads to an alternative definition of linearly compact vector spaces given by Drinfeld as linearly topologized complete Hausdorff spaces with the property that open subspaces have finite codimension [22]. Using results of Dieudonné, one can show that these two definitions are equivalent [21]. Compact vector spaces are also topological duals of discrete ones [22], [38]. We choose to take the duality viewpoint and follow Beilinson-Drinfeld in notation and conventions [22], [3]. So our compact vector spaces are topological duals $V^\star$ of discrete vector spaces V, where by a topological dual we mean the space of all continuous linear functionals on V. The topology on $V^\star$ is given by orthogonal complements of finite dimensional subspaces of V with respect to the canonical pairing [3].

Define

$$\mathcal{D} : \mathcal{M}_d(G) \to \mathcal{M}_c(G)$$

by

$$V \mapsto V^\star \text{ and } f \mapsto f^\star, \text{ where } f^\star : \varphi \mapsto \varphi \circ f.$$

We claim that this is a contravariant functor, which induces an anti-equivalence of categories. First, let us check that $\mathcal{D}$ is indeed a functor.

Lemma 3.5.1. *Suppose $R = \mathbb{F}$ is a field and let $(\pi, V) \in \mathcal{M}_d(G)$. Then $(\lambda, V^\star) \in \mathcal{M}_c(G)$, where $V^\star \coloneqq \{f : V \to \mathbb{F}\}$ is the space of all continuous linear functionals on V and λ is the action by left translations.*

Proof. We make $V^\star$ into a G-module by defining an action of G by left translation. More precisely, we have an action map $\phi : G \times V^\star \to V^\star$, given by $\phi : (\mathbf{g}, f) \mapsto \lambda_{\mathbf{g}} f$, where $\lambda_{\mathbf{g}} f : x \mapsto f(\pi(\mathbf{g}^{-1})x)$. We claim that ϕ is continuous.

Let $M^\star \subseteq V^\star$ be open. Suppose that $\lambda_{\mathbf{g}} f \in M^\star$, for some $\mathbf{g} \in G$ and $f \in V^\star$. By definition $M^\star \coloneqq \{f : V \to \mathbb{F} \mid f(m) = 0 \text{ for all } m \in M\}$, where $M \subseteq V$ is finite dimensional. Then $M = \mathbb{F}\langle m_1, .., m_n\rangle$, for some $m_i \in V, i = 1, ..., n$. Since $(\pi, V) \in \mathcal{M}_d(G)$, there exist open neighbourhoods K_i of 1_G, such that $\pi(\mathbf{k}_i)m_i = m_i$, for every $\mathbf{k}_i \in K_i$ and $i = 1, .., n$. Since G is a topological group, we can choose K_i to be symmetric. Let $K = \bigcap_{i=1}^n K_i$. This is an open symmetric neighbourhood of 1_G. Now consider $N \coloneqq \pi(\mathbf{g})M = \mathbb{F}\langle \pi(\mathbf{g})m_1, ..., \pi(\mathbf{g})m_n\rangle$. This is a finite dimensional subspace of V. Hence, $N^\star \coloneqq \{f : V \to \mathbb{F} \mid f(n) = 0 \text{ for all } n \in N\} \subseteq V^\star$ is open. Thus, $K \subseteq G$ and $N^\star \subseteq V^\star$ are both open and $\lambda(\mathbf{g}K)(f + N^\star) \subseteq M^\star$, finishing the proof. □

Lemma 3.5.1 shows that $\mathcal{D}$ maps objects to objects. Let us check it does the same on morphisms. Let $(\pi_1, V_1), (\pi_2, V_2) \in \mathcal{M}_d(G)$ and $f : V_1 \to V_2$ be a morphism in $\mathcal{M}_d(G)$. Then $f^\star : V_2^\star \to V_1^\star$ has the following properties:

1. $f^\star(\lambda_2(\mathbf{g})\varphi) = (\lambda_2(\mathbf{g})\varphi) \circ f$. Now

   $(\lambda_2(\mathbf{g})\varphi) \circ f : v_1 \mapsto \varphi(\pi_2(\mathbf{g}^{-1})f(v_1)) = \varphi(f(\pi_1(\mathbf{g}^{-1})v_1)) = \big(\lambda_1(\mathbf{g})f^\star(\varphi)(v_1)\big)$, where $\mathbf{g} \in G, v_1 \in V_1, \varphi \in V_2^\star$.

   Thus, $f^\star(\lambda_2(\mathbf{g})\varphi) = \lambda_1(\mathbf{g})f^\star(\varphi)(v_1)$ and $f^\star$ is G-linear.

2. Let $U^\star \subseteq V_1^\star$ be open. Then $f^\star(U^\star)^{-1} = \{\varphi \in V_2^\star \mid (\varphi \circ f)(U) = 0\}$, for $U \subseteq V_1$ of finite dimension. But since f is a linear map, then $f(U) \subseteq V_2$ is also a finite dimensional subspace. By definition of the topology on $V_2^\star$ it follows that $f^\star(U^\star)^{-1}$ is open and $f^\star$ is continuous.

Therefore, $\mathcal{D}$ is indeed a functor. It is clear from the definition of a compact vector space that $\mathcal{D}$ is bijective. Thus:

Lemma 3.5.2. *The functor $\mathcal{D}$ induces an anti-equivalence between $\mathcal{M}_d(G)$ and $\mathcal{M}_c(G)$. In particular, $\mathcal{D}$ maps products in $\mathcal{M}_d(G)$ to coproducts in $\mathcal{M}_c(G)$.*

Note that Kohlhase shows a special case of our result: he establishes that if G is a locally profinite group and (π, V) is a discrete representation of G over a field $\mathbb{F}$, then $V^\star$ has the structure of a compact module over the algebra $\Lambda(G)$, where $\Lambda(G)$ is a generalisation of the Iwasawa algebra of a compact group [38]. In particular, he obtains an anti-equivalence between $\mathcal{M}_d(G)$ and the category of pseudocompact $\Lambda(G)$-modules. However, dropping the restriction on the topology on G, we obtain an anti-equivalence between the category of discrete representations and the category of compact G-modules.

In the following example we use our observations about $\mathcal{D}$ to gain information about the cocontinuity of Res_H^G in $\mathcal{M}_c(G)$.

Example 3.5.3. Let $\{(\pi_i, V_i)\}_{i\in\mathcal{I}} \in \mathcal{M}_d(G)$, where $\mathcal{I}$ is an arbitrary set, and let $\{V_i\}_{i\in\mathcal{I}}$ be a collection of discrete vector spaces over a field $\mathbb{F}$. By Lemma 3.5.1 $(\lambda_i, V_i^\star) \in \mathcal{M}_c(G)$. By Freyd's Theorem to have a right adjoint to Res_H^G in $\mathcal{M}_c(G)$ we need

$$\mathrm{Res}_H^G\big(\bigoplus_{i\in\mathcal{I}}(\lambda_i, V_i^\star)\big) \cong \bigoplus_{i\in\mathcal{I}}\mathrm{Res}_H^G\big((\lambda_i, V_i^\star)\big), \tag{3.3}$$

where $\bigoplus$ denotes the coproduct. By Lemma 3.5.2

$$\mathrm{Res}_H^G\big(\bigoplus_{i\in\mathcal{I}}(\lambda_i, V_i^\star)\big) \cong \mathrm{Res}_H^G\big(\big(\big(\prod_{i\in\mathcal{I}}(\pi_i, V_i)\big)^{sm}\big)^\star\big). \tag{3.4}$$

However,

$$\bigoplus_{i\in\mathcal{I}} \mathrm{Res}_H^G \left((\lambda_i, V_i^\star)\right) \cong \left((\prod_{i\in\mathcal{I}}(\pi_i|_H, {}_HV_i))^{sm}\right)^\star. \tag{3.5}$$

By Theorem 3.3.4 $\mathrm{Res}_H^G \left(\left(\prod_{i\in\mathcal{I}}(\pi_i, V_i)\right)^{sm}\right) \cong \prod_{i\in\mathcal{I}} \left(((\pi_i|_H, {}_HV_i))^{sm}\right)$ if H is open. Thus, for $H \leqslant G$ open (3.3) holds and there is a well-defined functor

$$\mathrm{Ind}_H^G : \mathcal{M}_c(H) \to \mathcal{M}_c(G).$$

As explained in the beginning of this section, all linearly compact vector spaces are complete. This means that $\mathcal{M}_c(G)$ is a subcategory of $\mathcal{M}_{ltc}(G)$. In the next example we consider the coproduct $\bigoplus_{i\in\mathcal{I}}(\lambda_i, V_i^\star)$, for $(\lambda_i, V_i^\star) \in \mathcal{M}_c(G)$, in the category $\mathcal{M}_{ltc}(G)$. We aim to illustrate that in the case of topological vector spaces, if $H \leqslant G$ is not open, Ind_H^G is not always defined in $\mathcal{M}_{ltc}(G)$, too. We first wish to note that our definition of the compact topology on a topological vector space $V^\star$ using the duality viewpoint is equivalent to requiring that $V^\star$ is Hausdorff, linear, complete and such that every open subspace has finite codimension [22]. We are now ready for our example.

Example 3.5.4. For a collection $\{(\lambda_i, V_i^\star)\}_{i\in\mathcal{I}} \in \mathcal{M}_c(G)$ defined as in Example 3.5.3, consider their coproduct $\bigoplus_{i\in\mathcal{I}}(\lambda_i, V_i^\star)$ in $\mathcal{M}_{ltc}(G)$. As we will not need the maps λ_i explicitly, for simplicity we will just write $\bigoplus_{i\in\mathcal{I}} V_i^\star$ in this example. Keeping all other notation and conventions the same as in Section 3.4, recall that this is the coproduct in the category of $\mathbb{F}$-vector spaces with topology induced by the embedding

$$q : \bigoplus_{i\in\mathcal{I}} V_i^\star \hookrightarrow \prod_{(W_k,\tau_{W_k})} W_k.$$

Since we are considering the coproduct as an object of $\mathcal{M}_{ltc}(G)$, rather than $\mathcal{M}_c(G)$, some of the (W_k, τ_{W_k}) can be linearly topologized and complete, but not compact. We claim that regardless of whether this is the case, $\bigoplus_{i\in\mathcal{I}} V_i^\star$ is still an object of $\mathcal{M}_c(G)$. Let $N \leqslant \prod_{(W_k,\tau_{W_k})} W_k$ be an arbitrary open submodule. Then $N = \prod_{j\in\mathcal{J}} U_j \times \prod_{(W_k,\tau_{W_k}),k\notin\mathcal{J}} W_k$, where $U_j \leqslant W_j$ are open submodules and $\mathcal{J}$ is finite. We are interested in $q^{-1}(N)$. For $(v_i) \in \bigoplus_{i\in\mathcal{I}} V_i^\star$, write $q((v_i)) = (q_{W_j}((v_i)))$. For every i, for $v_i \in q^{-1}(N)$, v_i is such that $v_i \in q^i_{W_1}(U_1)^{-1} \cap \cdots \cap q^i_{W_l}(U_l)^{-1}$, where $l = |\mathcal{J}|$. Since the maps $q^i_{W_k}$ are continuous, $(q^i_{W_j})^{-1}(U_j)$ is open in $V_i^\star$ for every $i \in \mathcal{J}$. It is also clearly a vector subspace. Thus, $q^i_{W_1}(U_1)^{-1} \cap \cdots \cap q^i_{W_l}(U_l)^{-1}$ is an open subspace of $V_i^\star$ which is compact and so it is of finite codimension in $V_i^\star$. Thus, $q^{-1}(N)$ is of finite codimension in $\bigoplus_{i\in\mathcal{I}} V_i^\star$. As all open submodules of $\bigoplus_{i\in\mathcal{I}} V_i^\star$ correspond to inverse images of open submodules $N \leqslant \prod_{(W,\tau_W)} W$, the topology on $\bigoplus_{i\in\mathcal{I}} V_i^\star$ is compact. By the uniqueness of coproducts and Example 3.5.3 it follows that if H is not open, the restriction functor $\mathrm{Res}^G_H : \mathcal{M}_{ltc}(G) \to \mathcal{M}_{ltc}(H)$ is not cocontinuous.

3.5.2 The case of a general R

We move on to R-modules, where R is an associative ring with 1. We call an R-module V *compact* if V admits a linear complete topology with the additional property that if $U \subseteq V$ is an open submodule, then V/U is of finite length. Such modules are sometimes called *pseudocompact* [24], [34], [6]. We call a topological R-module V *linearly compact* if it is linearly topologized, Hausdorff, and such that every family of closed cosets in V has the finite intersection property [56]. Every compact module is linearly compact [34]. However, contrary to the case of vector spaces the notions of compactness and linear compactness are not equivalent. We denote by $\mathcal{M}_c(G)$ the category of all compact R-modules which admit a continuous action of the topological group G. We now construct products and coproducts in $\mathcal{M}_c(G)$.

Lemma 3.5.5. *Arbitrary products exist in $\mathcal{M}_c(G)$.*

Proof. A product of linearly compact R-modules is linearly compact with respect to the product topology [56, VII, 28.7]. Since every compact module is linearly compact, then the category of compact modules is closed under products.

By exactly the same argument as in Lemma 3.4.2 for an arbitrary collection $\{(\lambda_i, V_i)\}_{i\in\mathcal{I}}$ of objects of $\mathcal{M}_c(G)$ their product $V := \prod_{i\in\mathcal{I}} V_i$ as R-modules, endowed with the product topology, is a continuous G-module with respect to the componentwise action of G. ∎

We now wish to form coproducts in $\mathcal{M}_c(G)$. For an arbitrary collection $\{(\lambda_i, V_i)\}_{i\in\mathcal{I}} \in \mathcal{M}_c(G)$, we form the coproduct $V \coloneqq \bigoplus_{i\in\mathcal{I}} V_i$ as R-modules, with componentwise action of G, $\lambda = (\lambda_i)_{i\in\mathcal{I}}$. To define a topology $\mathcal{T}_V$ on V we mimic the procedure from Section 3.4. Let $W \in \mathcal{M}_c(G)$. Suppose there exists a surjective R-linear map $q_W : V \to W$ which commutes with the G-action, such that the maps $q_W^i : V_i \to W$, factoring through q_W, are continuous. The topology $\mathcal{T}_V$ on V is induced by the embedding

$$q : V \hookrightarrow \prod_{(W,\tau_W)} W, \quad v \mapsto (q_W(v))_{(W,\tau_W)}. \tag{3.6}$$

Lemma 3.5.6. *Let $\{(\lambda_i, V_i)\}_{i\in\mathcal{I}}$ be an arbitrary collection of objects of $\mathcal{M}_c(G)$. Their coproduct is (λ, V), where $\lambda = (\lambda_i)_{i\in\mathcal{I}}$ and $(V, \mathcal{T}_V)$ is the topological R-module described above.*

Proof. For simplicity, we abuse notation and write V_i for (λ_i, V_i), etc. By Lemma 3.5.5 $\prod_{(W,\tau_w)} W \in \mathcal{M}_{ltc}(G)$. Since $V \subseteq \prod_{(W,\tau_w)} W$, the topology on V is linear [40, 25.3]. Let $U \in \prod_{(W,\tau_w)} W$ be a basic open. Then $U = U_i \times \prod_{(W,\tau_w)} W$, where $U_i \subseteq W$ for some W, is an open submodule. By definition $q^{-1}(U)$ is open in V. But $q^{-1}(U) = \bigoplus_{i\in\mathcal{I}} q^{-1}(U) \cap V_i$. Since each $q_W^i : V_i \to W$ is continuous, it follows that $q^{-1}(U) \cap V_i$ is an open submodule of V_i. Hence, the quotient is of finite length. This implies that $V/q^{-1}(U)$ is also of finite length, showing that the topology $\mathcal{T}_V$ is compact. Moreover, V is linearly compact as every compact space is linearly compact. Thus, V is complete [56, VII, 28.6]. The map $G \times \prod_{(W,\tau_W)} W \to \prod_{(W,\tau_W)} W$ is continuous, and thus so is its restriction to a subspace, i.e., $V \in \mathcal{M}_c(G)$. As $\mathcal{M}_c(G) \subseteq \mathcal{M}_{ltc}(G)$ and the coproducts in the two categories are constructed in the same way, Lemma 3.4.3 implies that V satisfies the universal property of the coproduct in $\mathcal{M}_c(G)$, finishing the proof. $\square$

Having constructed products and coproducts in $\mathcal{M}_c(G)$, in order to establish existence of a left and a right adjoint to the restriction functor Res_H^G in $\mathcal{M}_c(G)$, we need to check whether it is continuous and cocontinuous.

Theorem 3.5.7. *Let G be a topological group and $H \leqslant G$ a closed subgroup. The restriction functor*

$$\mathrm{Res}_H^G : \mathcal{M}_c(G) \to \mathcal{M}_c(H)$$

has a left adjoint. Hence, we have a well-defined $\mathrm{Coind}_H^G : \mathcal{M}_c(H) \to \mathcal{M}_c(G)$*. It has a right adjoint if H is open.*

Proof. Since $\mathcal{M}_c(G)$ is a subcategory of $\mathcal{M}_{ltc}(G)$ and a submodule and a quotient module of a linearly compact module are linearly compact [40, 25.3], Lemma 3.4.4 implies that all morphisms of $\mathcal{M}_c(G)$ have kernels and cokernels. Thus, $\mathcal{M}_c(G)$ has equalizers and coequalizers of all pairs of morphisms. Since the products and coproducts in $\mathcal{M}_c(G)$ are the same as in $\mathcal{M}_{ltc}(G)$, the statement follows from Theorem 3.4.5. □

Chapter 4

Smooth representations

All material in this chapter is original. It is taken from a joint paper between Dmitriy Rumynin and the author of this thesis [32]. Whenever a known result is used it is appropriately referenced. The main reference for the preliminary definitions is Chapter 1 of Bushnell and Henniart [9]. However, they work with representations over the complex numbers and we work over an arbitrary field $\mathbb{F}$.

In this chapter we investigate the category of discrete representations of a topological group G over a field $\mathbb{F}$ in the special case when G is locally compact and totally disconnected. This category, as mentioned in Example 2.0.5, is denoted $\mathcal{M}(G)$, and is called the category of smooth representations of G. From now on G denotes a locally compact totally disconnected group and $\mathbb{F}$ is an arbitrary field. We will not impose any restrictions on the characteristic of $\mathbb{F}$ for now, however, later on in Chapters 4, 5 and 6 restrictions will apply. Whenever we need those, we will state them clearly.

4.1 Preliminary notions

Let us start with an explicit definition of a smooth representation. As it is a special case of a continuous representation where the topology on the vector space is discrete and the topology on the group is locally compact totally disconnected, we can rephrase the continuity condition in a more explicit manner:

Defintion 4.1.1. (cf. [4], [9]) A *smooth representation* of a locally compact totally disconnected group G is a pair (π, V), such that:

1. (representation) V is a topological $\mathbb{F}$-vector space with respect to the discrete topology, $\pi : V \to \mathrm{Aut}_{\mathbb{F}}(V)$ is a homomorphism,

2. (continuity) For every $v \in V$, $\mathrm{Stab}_G(v) = \{\mathbf{g} \in G \mid \pi(\mathbf{g})v = v\}$ is open in G.

Note that since the topology on G is locally compact totally disconnected, the continuity condition is equivalent to:

2'. For every $v \in V$ there exists a compact open subgroup $K_v \leqslant G$, such that $\pi(\mathbf{k})v = v$ for all $\mathbf{k} \in K_v$.

The morphisms in $\mathcal{M}(G)$ are defined in the same way as in $\mathcal{M}_d(G)$.

Let $(\pi, V) \in \mathcal{M}(G)$. Suppose that $U \leqslant V$ is a vector subspace. We say that U is a *G-subspace*, or a subrepresentation, if $\pi(G)U \subseteq U$. Thus, a smooth representation (π, V) of G is *irreducible* if V has no non-trivial G-subspaces. A smooth representation (π, V) of G is *semisimple* if V splits into a direct sum of its irreducible G-subspaces (cf. [9, 1.2]). Let $A \leqslant G$ be a subgroup. We call A *central*, if $A \subseteq Z(G)$, where $Z(G)$ is the centre of G.

Now fix a closed central subgroup $A \leqslant G$, which could be trivial. A simple representation of A is just a simple $\mathbb{F}$-representation of the group algebra $\mathbb{F}A$. Hence, it is determined by a field extension $\widetilde{\mathbb{F}} \supseteq \mathbb{F}$ and a character $\chi : A \to \widetilde{\mathbb{F}}^\times$ ($\widetilde{\mathbb{F}}^\times$ denotes the multiplicative group of $\widetilde{\mathbb{F}}$), such that $\widetilde{\mathbb{F}}$ is generated as an $\mathbb{F}$-algebra by the image of χ. We denote this representation by $\widetilde{\mathbb{F}}_\chi$ and the set of such characters by $\mathrm{Irr}(\mathbb{F}A)$. We want to study A-semisimple smooth representations of G.

Defintion 4.1.2. An *A-semisimple smooth representation* of G is a smooth representation (π, V) which is semisimple as a representation of A. By $\mathcal{M}_A(G)$ we denote the abelian category of A-semisimple smooth representations of G. For each character $\chi \in \mathrm{Irr}(\mathbb{F}A)$ we denote by $\mathcal{M}_{A,\chi}(G)$ the full subcategory of $\mathcal{M}_A(G)$ of those representations that are direct sums of $\widetilde{\mathbb{F}}_\chi$ as representations of A.

4.2 Induced and restricted representations

Let $H \leqslant G$ be a closed subgroup. Then H is also locally compact and totally disconnected. We know from Chapter 3 there exists a functor $\mathrm{Res}_H^G : \mathcal{M}(G) \to \mathcal{M}(H)$ called the restriction functor. Keeping the notation of Section 3.3 we have the following:

- Res_H^G has a right adjoint, Ind_H^G. For any $(\sigma, W) \in \mathcal{M}(H)$, we have $\mathrm{Ind}_H^G((\sigma, W)) = \mathrm{Ind}_H^G(\sigma) = (\rho, \widetilde{W})$, called the representation *smoothly induced by* σ. Recall that $\widetilde{W}$ is the $\mathbb{F}$-vector space of all functions $f : G \to W$, which are H-equivariant and satisfy the property that for each f, there exists a compact open subgroup $K_f \leqslant G$, such that $f(\mathbf{gk}) = f(\mathbf{g})$, for all $\mathbf{g} \in G$ and $\mathbf{k} \in K_f$. The action ρ is by right translations, i.e., $\rho(\mathbf{g})f : \mathbf{x} \mapsto f(\mathbf{xg})$, for all $\mathbf{x}, \mathbf{g} \in G$ (cf. Section 3.3.2).

- Whenever H is also an open subgroup, Res_H^G has a left adjoint, given by *compact induction* $c - \mathrm{Ind}_H^G$. Recall that for $(\sigma, W) \in \mathcal{M}(H)$, we have $c - \mathrm{Ind}_H^G((\sigma, W)) = \mathrm{Ind}_H^G(\sigma) = (\rho, \widetilde{W})$, where σ is as above, and all $f \in \widetilde{W}$ satisfy the additional condition that they are compactly supported modulo H (cf. Example 3.3.5).

There is also a notion of *algebraic induction*, denoted $a - \mathrm{Ind}_H^G$. Let $(\sigma, W) \in \mathcal{M}(H)$. The $\mathbb{F}H$-module $\mathbb{F}G \underset{\mathbb{F}H}{\otimes} W$ becomes an $\mathbb{F}G$-module by setting $\mathbf{g}(\mathbf{g}' \otimes w) = \mathbf{gg}' \otimes w$ for $\mathbf{g}, \mathbf{g}' \in G$, $w \in W$. If $H \leqslant G$ is open, $a - \mathrm{Ind}_H^G(\sigma)$ is a smooth representation of G. Moreover, the map $\varphi : \mathbb{F}G \underset{\mathbb{F}H}{\otimes} W \to \mathrm{Fun}_H(G_H, {}_HW)$ given by

$$\mathbf{g} \otimes w \mapsto (f : \mathbf{gh}^{-1} \mapsto \mathbf{h}w), \quad \mathbf{g} \in G, \mathbf{h} \in H, w \in W,$$

is an isomorphism from $a - \mathrm{Ind}_H^G(\sigma)$ to $c - \mathrm{Ind}_H^G(\sigma)$.

We claim that the functors $\mathrm{Ind}_H^G, c{-}\mathrm{Ind}_H^G, a{-}\mathrm{Ind}_G^H$ preserve A-semisimplicity.

Lemma 4.2.1. *Let G be a locally compact totally disconnected group. Suppose $H \geqslant A$ is a subgroup of G, closed and compact modulo A. The following hold:*

1. Ind_H^G *and* $c-\mathrm{Ind}_H^G$ *define functors from* $\mathcal{M}_A(H)$ *(or* $\mathcal{M}_{A,\chi}(H)$*) to* $\mathcal{M}_A(G)$ *(*$\mathcal{M}_{A,\chi}(G)$ *correspondingly).*

2. *In the case when* H *is also open,* $a-\mathrm{Ind}_H^G$ *also defines a functor from* $\mathcal{M}_A(H)$ *(or* $\mathcal{M}_{A,\chi}(H)$*) to* $\mathcal{M}_A(G)$ *(*$\mathcal{M}_{A,\chi}(G)$ *correspondingly).*

Proof. Let $(\sigma, W) \in \mathcal{M}_A(H)$. As in Section 3.3.2, let $\widehat{W}$ denote the $\mathbb{F}$-vector space $\widehat{W}$ of all H-equivariant functions $f : G \to W$ and $\widetilde{W} \subseteq \widehat{W}$ the $\mathbb{F}$-vector subspace of all smooth functions. We have the homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(\widehat{W})$ given by $[\rho(\mathbf{g})f](\mathbf{g}') = f(\mathbf{g}'\mathbf{g})$, for $\mathbf{g}, \mathbf{g}' \in G$ and $f \in \widehat{W}$. For $f \in \widehat{W}$ and $\mathbf{a} \in A$, we see that $[\rho(\mathbf{a})f](\mathbf{g}) = f(\mathbf{ga}) = f(\mathbf{ag}) = \sigma(\mathbf{a})f(\mathbf{g})$, for all $\mathbf{g} \in G$. Writing $W = \oplus_i W_i$ as a direct sum of simple A-modules $W_i = \widetilde{\mathbb{F}}_{\chi_i}$, we can present $f = \sum_i f_i$ as a sum of A-equivariant smooth functions $f_i : G \to W_i$, so that $[\rho(\mathbf{a})f](\mathbf{g}) = \sum_i \sigma(\mathbf{a})f_i(\mathbf{g}) = \sum_i [\chi_i(\mathbf{a})f_i](\mathbf{g})$. This proves that $(\rho, \widehat{W})$ is A-semisimple (but not smooth). Its submodule $(\rho, \widetilde{W})$ is smooth (by construction) and also A-semisimple. Hence, $(\rho, \widetilde{W}) \in \mathcal{M}_A(G)$. Having proved the first statement, the second statement easily follows, as $c - \mathrm{Ind}_H^G(\sigma)$ is a subspace of $\mathrm{Ind}_H^G(\sigma)$ of all compactly supported modulo H functions, and for H open, $c - \mathrm{Ind}_H^G \cong a - \mathrm{Ind}_H^G$. □

4.3 The Haar Integral on G

Following Bushnell and Henniart [9, 1.3], we define a measure and thus integration for a locally compact totally disconnected group G. Let $C_c^\infty(G)$ denote the space of all functions $\theta : G \to \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, which are smooth and with compact support. Note that due to the topology on G the sets of continuous, smooth and locally constant functions coincide. G acts on $C_c^\infty(G)$ on the left $\lambda : G \to \mathrm{Aut}_{\mathbb{C}}(C_c^\infty(G))$, $\mathbf{g} \mapsto \lambda_{\mathbf{g}}$, and on the right $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(C_c^\infty(G))$, $\mathbf{g} \mapsto \rho_{\mathbf{g}}$, where:

$$\lambda_{\mathbf{g}}\Theta : x \mapsto \Theta(\mathbf{g}^{-1}x), \text{ for } \mathbf{g} \in G, \Theta \in C_c^\infty(G),$$

$$\rho_{\mathbf{g}}\Theta : x \mapsto \Theta(x\mathbf{g}), \text{ for } \mathbf{g} \in G, \Theta \in C_c^\infty(G).$$

Moreover, $(\lambda, C_c^\infty(G)), (\rho, C_c^\infty(G)) \in \mathcal{M}(G)$.

Defintion 4.3.1. [9, 1.3.1] A *left Haar integral* on G is a non-zero linear functional $I : C_c^\infty(G) \to \mathbb{C}$, such that:

1. $I(\lambda_{\mathbf{g}}\Theta) = I(\Theta)$ for all $\mathbf{g} \in G$ and $\Theta \in C_c^\infty(G)$,
2. $I(\Theta) \geqslant 0$ for all $\Theta \in C_c^\infty(G)$.

Similarly define the *right Haar integral.*

A left (right) Haar integral on G always exists and is unique up to a scalar. In particular, if I and I' are left (right) Haar integrals on G, then there exists a $c \in \mathbb{R}_{>0}$, such that $I' = cI$ [9, 1.3.1, Proposition].

There also exists a left (right) translation invariant Haar measure defined on the Borel algebra of G, called the *left (right) Haar measure on* G denoted μ. The relation with the Haar integral is

$$I(\Theta) := \int_G \Theta(\mathbf{g})\mu(d\mathbf{g}),$$

where $\Theta \in C_c^\infty(G)$. For any $\mathbf{h} \in G$, the functional

$$I'(\Theta) := \int_G \Theta(\mathbf{gh})\mu(d\mathbf{g}), \quad \mathbf{g}, \mathbf{h} \in G$$

is also a left Haar integral. Hence, there is a $c_{\mathbf{h}} \in \mathbb{R}_{>0}$, such that $I'(\Theta) = c_{\mathbf{h}}I(\Theta)$. This defines a group homomorphism

$$\Delta : G \to \mathbb{R}_{>0}^\times, \text{ given by } \Delta : \mathbf{g} \mapsto c_{\mathbf{h}}$$

called the *modular function* [9, 1.3.3]. Note that this is a character of G.

Defintion 4.3.2. [9, 3.1] A locally compact totally disconnected group G is called *unimodular* if any left Haar measure on G is also a right Haar measure.

Note that a group is unimodular if and only if the modular function is identically 1 [9, 1.3.3].

Suppose $K \leqslant G$ is a compact subgroup and μ is a Haar measure on G. Then

$$\mu(K) = I(X_K),$$

where I is a left (right) Haar integral on G and X_K is the characteristic function on K [9, 1.3.3]. Since the Haar measure on compact sets is always finite, we can choose a left Haar measure μ_K on G, such that $\mu_K(K) = 1$. The measure μ_K is called *normalised*.

Let μ_K be a normalised left Haar measure on G. Let $\mathcal{I}$ be the set of indices $|K : C|$ of all compact open subgroups $C \leqslant K$. Let $\mathbb{Z}_{(K)}$ be the ring of fractions on $\mathbb{Z}$ obtained by inverting all numbers $n \in \mathcal{I}$.

Lemma 4.3.3. *[32] (cf. [55, Lemma 2.4]) If $A \subseteq G$ is a Borel set, then $\mu_K(A) \in \mathbb{Z}_{(K)} \cup \{\infty\}$. Moreover, $\Delta(\mathbf{x}) \in \mathbb{Z}_{(K)}$ for all $\mathbf{x} \in G$.*

Proof. The topology of G admits a basis at 1_G consisting of compact open subgroups [30, II.7.7]. If N be a compact open subgroup, then it is commensurable to K, i.e., $|K : N \cap K| \leqslant \infty$ and $|N : N \cap K| \leqslant \infty$. Moreover, as K and N are compact, so is $K \cap N$, and thus $|K : N \cap K| \in \mathcal{I}$. Hence

$$\mu_K(N) = \frac{|N : (N \cap K)|}{|K : (N \cap K)|} \in \mathbb{Z}_{(K)}.$$

Since A is a disjoint union of left cosets of various compact open subgroups, $\mu_K(A) \in \mathbb{Z}_{(K)} \cup \{\infty\}$. Finally, $\Delta(\mathbf{x}) = \mu_K(K\mathbf{x}) \in \mathbb{Z}_{(K)}$. □

Our next result is useful as a unimodularity criterion.

Proposition 4.3.4. *Consider a compact open subgroup $H \leqslant G$ and $\mathbf{x} \in G$. Then*

$$\Delta(\mathbf{x}) \cdot |H : H \cap \mathbf{x}^{-1} H \mathbf{x}| = |H : H \cap \mathbf{x} H \mathbf{x}^{-1}|.$$

Proof. For a compact open subgroup H, $\mu(H) = \mu_K(H)$ is finite [9]. Thus, it suffices to observe that

$$\Delta(\mathbf{x})\mu(H) = \mu(H\mathbf{x}) = \mu(\mathbf{x}^{-1}H\mathbf{x}) = \frac{|\mathbf{x}^{-1}H\mathbf{x} : H \cap \mathbf{x}^{-1}H\mathbf{x}|}{|H : H \cap \mathbf{x}^{-1}H\mathbf{x}|} \cdot \mu(H) =$$

$$= \frac{|H : H \cap \mathbf{x}H\mathbf{x}^{-1}|}{|H : H \cap \mathbf{x}^{-1}H\mathbf{x}|} \cdot \mu(H).$$

□

If $K \leqslant G$ is compact, then K is a profinite group. Recall that the order $|K|$ of a profinite group K is a supernatural number $\prod_p p^{n_p}$, where p runs through the set of all primes and $n_p \in \{0, 1, \ldots, \infty\}$, that is the least common multiple of orders of K/H for various open subgroups $H \leqslant K$ [47, 2.3]. With this in mind, we make the following definition:

Defintion 4.3.5. Let $K \leqslant G$ be a compact subgroup. Suppose the field $\mathbb{F}$ is equipped with the discrete topology. We say that the field (or its characteristic) is K-*modular*, if $\mathrm{char}(\mathbb{F})$ divides the order $|K|$. Similarly, it is K-*ordinary*, if $\mathrm{char}(\mathbb{F})$ does not divide $|K|$.

Example 4.3.6. Suppose $\mathrm{char}(\mathbb{F}) = 0$ and let G be any locally compact totally disconnected group. Then $\mathbb{F}$ is K-ordinary for any compact subgroup $K \leqslant G$. Thus, all the results in Chapter 4, 5, 6 and 7 hold over fields of characteristic zero, i.e., we can always take $\mathbb{F}$ to be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$ (or more generally any non-archimedean local field), or a number field.

Example 4.3.7. Suppose $\mathrm{char}(\mathbb{F}) > 0$. Let $G = \widehat{G_{\mathfrak{D}}(\mathbb{F}_q)}$ be the local pro-p completion of the Kac-Moody group $G_{\mathfrak{D}}(\mathbb{F}_q)$ over the field of q elements $\mathbb{F}_q$, where q is a power of a prime p (cf. Chapter 7 for more details). Let $K = \widehat{U}$ be the completion of the subgroup U, which is generated by the root subgroups U_α, for $\alpha \in \Phi_+$. Then K is a Sylow pro-p subgroup of G, in particular, the order of K is a power of p. Thus, $\mathbb{F}$ is K-ordinary if $\mathrm{char}(\mathbb{F}) \neq p$. If $\mathrm{char}(\mathbb{F}) = p$, $\mathbb{F}$ is K-modular.

Example 4.3.8. Let $K \leqslant G$ be compact subgroup with order $\prod_p p^{n_p}$ as above. Suppose $\mathrm{char}(\mathbb{F}) \neq 0$, i.e., $\mathrm{char}(\mathbb{F}) = p$, for some prime p. In general, $\mathbb{F}$ will be

K-modular, if $n_p > 0$, where n_p is the power of p in the factorisation of the order of K, and K-ordinary if $n_p = 0$.

Let $K \leqslant G$ be a compact subgroup and μ_K be the normalised at K left Haar measure of G. The reason we are interested in $\mathbb{F}$ being a K-ordinary field is that if this is the case, then we can think of μ_K and, consequently, the modular function Δ, taking values in $\mathbb{F}^\times$. More precisely, if $\mathbb{F}$ is K-ordinary, there is a natural ring homomorphism $\mathbb{Z}_{(K)} \to \mathbb{F}$ and, thus, by Lemma 4.3.3 we may think that the measure μ_K and the modular function Δ take values in $\mathbb{F}^\times$. Hence, given a compactly supported smooth function $\Theta : G \to \mathbb{F}$, one can compute its Haar integral $\int_G \Theta(\mathbf{x})\mu_K(d\mathbf{x}) \in \mathbb{F}$. This will be useful for us when studying projectivity.

Using the Haar integral on G we can prove the following useful fact:

Lemma 4.3.9. *Let H be a subgroup of G, compact modulo A. Suppose that the field $\mathbb{F}$ is H/A-ordinary. Then the categories $\mathcal{M}_{A,\chi}(H)$ and $\mathcal{M}_A(H)$ are semisimple.*

Proof. Let $(\rho, V) \in \mathcal{M}_A(H)$. Then by definition V is A-semisimple and hence can be decomposed as $V = \bigoplus\limits_\chi V_\chi$ with $V_\chi = \{v \in V \mid \rho(\mathbf{a})v = \chi(\mathbf{a})v \text{ for all } \mathbf{a} \in A\}$, where $\chi \in \mathrm{Irr}(\mathbb{F}A)$. In other words, $\mathcal{M}_A(H) = \bigoplus\limits_\chi \mathcal{M}_{A,\chi}(H)$, so it is enough to prove the statement for $\mathcal{M}_{A,\chi}(H)$.

Let $V \in \mathcal{M}_{A,\chi}(H)$. Then V is an $\widetilde{\mathbb{F}}$-vector space with an $\widetilde{\mathbb{F}}$-linear H-action. Let $v \in V$. By smoothness there exists a compact open subgroup K_v of H, such that $\rho(\mathbf{k})v = v$ for all $\mathbf{k} \in K_v$. Let $V' := \langle Hv \rangle_{\widetilde{\mathbb{F}}}$. As both H/A and K_v are compact, H/AK_v is compact. It is also discrete: for $\dot{x} \in H/AK_v$, $\dot{x} = xAK_v$, which is the homeomorphic image of AK_v, and AK_x is open as it is a product of an open and closed set [30]. Thus, every element of H/AK_v is open, hence, H/AK_v is discrete. Since H/AK_v is both compact and discrete, it is finite. Thus, V' is a finite dimensional $\widetilde{\mathbb{F}}$-subspace of V.

We want to show that V is H-semisimple. It suffices to find a direct $\widetilde{\mathbb{F}}H$-complement in V of a finite dimensional H-submodule W. Let $p : V \to W$ be an $\widetilde{\mathbb{F}}$-linear projection. Since W is finite dimensional, there is a basis $e_1, \ldots, e_n$ of W and we can write $p(v) = \sum\limits_{i=1}^n p_i(v)e_i$, for some linear functionals $p_i : V \to \widetilde{\mathbb{F}}$.

Pick a section $s : \mathbf{x} \mapsto \dot{\mathbf{x}}$ of the quotient homomorphism $q : H \to H/A$. Let μ be a Haar measure on H/A. It exists since A is abelian (and thus unomodular), so $\Delta_{H|A} = \Delta_A \equiv 1$. Since we assumed that $\mathbb{F}$ is H/A-ordinary, it follows that one can compute Haar integrals over H/A with values in $\mathbb{F}$ (follows from Lemma 4.3.3 and the discussion before this lemma). Define a map $\widehat{p} : V \to W$ by

$$\widehat{p}(v) := \int\limits_{H/A} \dot{\mathbf{x}}^{-1} p(\dot{\mathbf{x}} v) \mu(d\mathbf{x}).$$

The map $\widehat{p}$ is well-defined: write $\dot{\mathbf{x}}^{-1} p(\dot{\mathbf{x}} v) = \sum\limits_i \sum\limits_j \psi_{ij}(\mathbf{x}^{-1}) \varphi_j(\mathbf{x}) e_i$ for some smooth functions $\psi_{ij}, \varphi_i : H \to \widetilde{\mathbb{F}}$, then integrate the functions.

Clearly, $\widehat{p}$ is a well-defined $\widetilde{\mathbb{F}}$-linear projection. Let us verify that it is also H-linear, i.e., that for all $\mathbf{y} \in H, v \in V$, $\widehat{p}(\mathbf{y}v) = \mathbf{y}\widehat{p}(v)$. Let $\overline{\mathbf{y}} = \mathbf{y}A \in H/A$. To use the standard argument we need a change of variable $\mathbf{z} = \mathbf{x}\overline{\mathbf{y}}$. Observe that $\mu(d\mathbf{z}) = \Delta(\bar{y})\mu(d\mathbf{x})$, but since H/A is compact, it is unimodular and so $\mu(d\mathbf{z}) = \mu(d\mathbf{x})$. Then $\dot{\mathbf{x}}\mathbf{y} = \mathbf{a}_{\mathbf{x}}\dot{\mathbf{z}}$ for some element $\mathbf{a}_{\mathbf{x}} \in A$ depending on $\mathbf{x}$ (we think that $\mathbf{y}$ is fixed). Furthermore, $\dot{\mathbf{x}}^{-1} = \mathbf{a}_{\mathbf{x}}^{-1}\mathbf{y}\dot{\mathbf{z}}^{-1}$ and

$$\widehat{p}(\mathbf{y}v) = \int\limits_{H/A} \dot{\mathbf{x}}^{-1} p(\dot{\mathbf{x}}\mathbf{y}v)\mu(d\mathbf{x}) = \int\limits_{H/A} \mathbf{a}_{\mathbf{x}}^{-1}\mathbf{y}\dot{\mathbf{z}}^{-1} p(\mathbf{a}_{\mathbf{x}}\dot{\mathbf{z}}v)\mu(d\mathbf{z}) = \mathbf{y}\widehat{p}(v).$$

The last equality holds because $\mathbf{a}_{\mathbf{x}}$ acts via the scalar $\chi(\mathbf{a}_{\mathbf{x}}) \in \widetilde{\mathbb{F}}$ and p is $\widetilde{\mathbb{F}}$-linear. This yields a decomposition $V = W \oplus \ker(\widehat{p})$, finishing the proof. □

If A is trivial and hence H is compact, then the category $\mathcal{M}(H)$ of smooth representations of H is semisimple.

4.4 The Hecke algebra of a locally compact totally disconnected group

Again we suppose that G is a locally compact totally disconnected group and $\mathbb{F}$ is an arbitrary field. We do not put restrictions on char$(\mathbb{F})$ yet, we will impose those when they are needed. The aim of this section is to define the *Hecke algebra* of G over $\mathbb{F}$ and to investigate some of its properties. We also explain the notion of a smooth (also called non-degenerate) modules over the Hecke algebra. Both the Hecke algebra and the category of its smooth modules will play a crucial role in our investigation of projective resolutions of smooth representations of G. In particular, by establishing certain equivalences of categories in the next section (Section 4.5) between $\mathcal{M}_A(G)$, $\mathcal{M}_{A,\chi}(G)$ and certain categories of smooth modules over the Hecke algebra, we see that projective resolutions of objects in the categories $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$ exist since the categories they are equivalent to on the Hecke algebra side have enough projectives. Let us start by defining the Hecke algebra of G.

Denote by $C_c^\infty(G,\mathbb{F})$ the $\mathbb{F}$-vector space of smooth functions $f: G \to \mathbb{F}$ with compact support, where $\mathbb{F}$ is our field.

Defintion 4.4.1. (cf. [55], [9]) The Hecke algebra of G over $\mathbb{F}$, denoted $\mathcal{H}(G,\mathbb{F},\mu_K)$, is the algebra $(C_c^\infty(G,\mathbb{F}),+,\star)$. More precisely, it is the algebra obtained from the $\mathbb{F}$-vector space $C_c^\infty(G,\mathbb{F})$, where the multiplication $\star$ is given by the convolution product. This is defined as follows: for $\Psi,\Theta \in C_c^\infty(G,\mathbb{F})$

$$\Psi \star \Theta(\mathbf{x}) = \int_G \Psi(\mathbf{y})\Theta(\mathbf{y}^{-1}\mathbf{x})\mu_K(d\mathbf{y}).$$

Notice that at first glance the Hecke algebra depends on three things - the locally compact totally disconnected group G, a compact subgroup $K \leqslant G$ where we normalise the Haar measure, i.e., μ_K is a left Haar measure on G, such that $\mu_K(K) = 1$, and, of course, the field $\mathbb{F}$. However, since we require the convolution product to take values in the field $\mathbb{F}$, we would need $\mathbb{F}$ to be K-ordinary for our definition to make sense (again we use Lemma 4.3.3 and the discussion before Lemma 4.3.9). Thus, we observe the following:

Lemma 4.4.2. *The Hecke algebra as defined above exists if there exists a compact subgroup $K \leqslant G$, such that $\mathbb{F}$ is K-ordinary. If two such compact subgroups $K, K' \leqslant G$ exist, the corresponding Hecke algebras $\mathcal{H}(G,\mathbb{F},\mu_K)$ and $\mathcal{H}(G,\mathbb{F},\mu_{K'})$ are isomorphic.*

Proof. The first statement follows from the paragraph above. Now let us prove the second. Suppose there exist subgroups $K, K' \leqslant G$, such that K and K' are compact and $\mathbb{F}$ is both K-ordinary and K'-ordinary. Let μ_K and $\mu_{K'}$ be the left Haar measures on G which are normalised at K and K' respectively. Then, by the properties of Haar measures we know that $\mu_K = \alpha\mu'_K$, for some scalar $\alpha \in \mathbb{F}^{\times}_{>0}$. Let $f : \mathcal{H}(G, \mathbb{F}, \mu_K) \to \mathcal{H}(G, \mathbb{F}, \mu_{K'})$ be an $\mathbb{F}$-algebra homomorphism. Then

$$f(\Psi \star \Theta) = f(\Psi) \star' f(\Theta), \quad \text{where} \quad f(\Psi) = \alpha\Psi.$$

Thus, $\mathcal{H}(G, \mathbb{F}, \mu_K)$ and $\mathcal{H}(G, \mathbb{F}, \mu_{K'})$ are isomorphic. $\square$

Recall that:

Defintion 4.4.3. (cf. [46, I.1.2]) Let $\mathfrak{A}$ be an algebra, a priori not unital. $\mathfrak{A}$ is called an *idempotented algebra*, if for each finite set $\{a_i\} \in \mathfrak{A}$, there exists an idempotent $e \in \mathfrak{A}$, such that $a_i = ea_ie$, for all i.

Defintion 4.4.4. (cf. [46, I.1.2]) Let $\mathfrak{A}$ be an idempotented algebra and $\mathrm{Idem}(\mathfrak{A})$ be a family of idempotents of $\mathfrak{A}$. We say that $\mathrm{Idem}(\mathfrak{A})$ *approximates the identity* if

$$\mathfrak{A} = \bigcup_{e \in \mathrm{Idem}(\mathfrak{A})} e\mathfrak{A}e.$$

The Hecke algebra of a locally compact totally disconnected group is not always unital (cf. [9, 55]). It contains an identity only if G is discrete (cf. [9, 55]). However, it is always an idempotented algebra (cf. [9, 55]). Let $K \leqslant G$ be a compact subgroup, such that $\mathbb{F}$ is K-ordinary. Let us explicitly describe the family of idempotents in $\mathcal{H}(G, \mathbb{F}, \mu_K)$ which approximate the identity.

Let $U \subseteq G$ be a compact open subset. Define a function $\Lambda_U \in \mathcal{H}(G, \mathbb{F}, \mu_K)$ by

$$\Lambda_U(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \notin U, \\ 1/\mu_K(U), & \mathbf{x} \in U. \end{cases}$$

The functions Λ_U satisfy $\Lambda_U \star \Lambda_U = \Lambda_U$, i.e., they are idempotents in $\mathcal{H}(G, \mathbb{F}, \mu_K)$

[9, 1.4].

Lemma 4.4.5. *(cf. [9, 1.4]) Let $\mathcal{B}$ be a basis at 1_G for the topology of G consisting of all compact open subgroups. Then the family $\{\Lambda_C\}_{C\in\mathcal{B}}$ forms a family of idempotents in $\mathcal{H}(G, \mathbb{F}, \mu_K)$ approximating the identity.*

Recall from Section 4.3 that the vector space $C_c^\infty(G)$ of $\mathbb{C}$-valued compactly supported functions on G has a structure of a left and a right G-module. As $(\mathcal{H}, \mathbb{F}, \mu_K)$ is defined as the algebra $(C_c^\infty(G, \mathbb{F}), +, \star)$, it follows that $(\mathcal{H}, \mathbb{F}, \mu_K)$ is also a left and a right G-module. Moreover, the actions are commuting so it has the structure of a $G - G$-bimodule.

4.5 Equivalence of categories

We continue with notation and conventions as in the previous section. In particular, G is a locally compact totally disconnected group, $K \leqslant G$ is a compact subgroup, such that the field $\mathbb{F}$ is K-ordinary, and $\mathcal{H}(G, \mathbb{F}, \mu_K)$ is the Hecke algebra of G. Let us start with an investigation of modules over the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$.

Defintion 4.5.1. (cf. [9]) Let $(M, *)$ be an $\mathcal{H}(G, \mathbb{F}, \mu_K)$-module. M is called *smooth* if $\mathcal{H}(G, \mathbb{F}, \mu_K) * M = M$. This is equivalent to saying that for every $m \in M$ there exists a compact open subgroup C of G, such that $\Lambda_C * m = m$. All smooth $\mathcal{H}(G, \mathbb{F}, \mu_K)$-modules form a category which we denote by $\mathcal{M}(\mathcal{H})$.

Sometimes smooth $\mathcal{H}(G, \mathbb{F}, \mu_K)$-modules are called *non-degenerate* [46]. The next theorem is one of the crucial results for our investigation of projective resolutions in $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$. Bushnell-Henniart prove it for the case of $\mathrm{char}(\mathbb{F}) = 0$ [9, 1.4.2] (in which case the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ as defined in the previous section always exists) and Vigneras proves it in positive characteristic [55, I.4.4].

Theorem 4.5.2. *[55, I.4.4](cf. [9, 1.4.2]) If $\mathcal{H}(G, \mathbb{F}, \mu_K)$ exists (i.e., if there exists* *a compact open subgroup K, such that the field $\mathbb{F}$ is K-ordinary), then the functor*

$$\mathcal{F} : \mathcal{M}(G) \to \mathcal{M}(\mathcal{H}),\ \mathcal{F}\Big((\pi, V)\Big) := (\varpi, V),\ \varpi(\Theta)v = \int_G \Theta(\mathbf{g})\pi(\mathbf{g})v\mu(d\mathbf{g})$$

for all $\Theta \in \mathcal{M}(\mathcal{H})$, $v \in V$, is an equivalence of categories.

Recall that $A \leqslant G$ is a closed central subgroup for which we have the full subcategories $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$ of the category $\mathcal{M}(G)$ of smooth representations of G. Thus, using this functor $\mathcal{F}$ above we can define their corresponding subcategories on the $\mathcal{M}(\mathcal{H})$ side. More precisely, let

$$\mathcal{M}_{A,\chi}(\mathcal{H}) := \overline{\mathcal{F}(\mathcal{M}_{A,\chi}(G))},\ \text{and}\ \mathcal{M}_A(\mathcal{H}) := \overline{\mathcal{F}(\mathcal{M}_A(G))},$$

i.e., $\mathcal{M}_A(\mathcal{H})$ and $\mathcal{M}_{A,\chi}(\mathcal{H})$ are the full subcategories of $\mathcal{M}(\mathcal{H})$ with objects isomorphic to the objects $\mathcal{F}\big((\pi, V)\big)$, where (π, V) lies in $\mathcal{M}_A(G)$ or $\mathcal{M}_{A,\chi}(G)$ respectively. Thus, we have:

Corollary 4.5.3. *If there exists a compact open subgroup $K \leqslant G$, such that $\mathbb{F}$ is K-ordinary (i.e., $\mathcal{H}(G, \mathbb{F}, \mu_K)$ exists), then $\mathcal{M}_A(G)$ is equivalent to $\mathcal{M}_A(\mathcal{H})$.*

Corollary 4.5.3 will be very important in our investigation of projective resolutions in $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$.

Remark 4.5.4. Note that if $\mathrm{char}(\mathbb{F}) = 0$, then $\mathbb{F}$ is always K-ordinary for any compact subgroup $K \leqslant G$. Thus, the existence of $\mathcal{H}(G, \mathbb{F}, \mu_K)$ is really only conditional when $\mathrm{char}(\mathbb{F}) > 0$.

Chapter 5

Projective resolutions in $\mathcal{M}(G)$

To the best of my knowledge the material presented in this chapter is original. It appears in a joint paper of Dmitriy Rumynin and the author of this thesis [32]. If a known result is used it is appropriately referenced. In Section 5.2 we follow Gelfand and Manin in notation and conventions on simplicial sets [27]. Throughout the chapter G denotes a locally compact totally disconnected group, $A \leqslant G$ is a closed central subgroup, $K \leqslant G$ is some compact subgroup, such that the field $\mathbb{F}$ is K-ordinary and $\mathcal{H}(G, \mathbb{F}, \mu_K)$ is the Hecke algebra of G.

5.1 Some results on projective modules

Recall that $\mathrm{Irr}(\mathbb{F}A)$ is the set of all characters of the closed central subgroup $A \leqslant G$ of the form $\chi : A \to \widetilde{\mathbb{F}}^\times$, where $\widetilde{\mathbb{F}} \supseteq \mathbb{F}$ is a field extension of $\mathbb{F}$, such that $\widetilde{\mathbb{F}}$ is generated by $\mathrm{im}(\chi)$ as an $\mathbb{F}$-algebra.

Let $(\pi, V) \in \mathcal{M}(G)$ and take $\chi \in \mathrm{Irr}(\mathbb{F}A)$. Consider the module

$$V_{A,\chi} \coloneqq \widetilde{\mathbb{F}} \otimes_{\mathbb{F}} V/V', \text{ where } V' = \langle \{1 \otimes \pi(\mathbf{a})v - 1 \otimes \chi(\mathbf{a})v\}_{v \in V, \mathbf{a} \in A} \rangle_{\widetilde{\mathbb{F}}}.$$

Then $V_{A,\chi} \cong \widetilde{\mathbb{F}} \otimes_{\mathbb{F}A} V$ and also $V_{A,\chi}$ is an object of $\mathcal{M}_{A,\chi}(G)$. We call this module the *(skew) coinvariants* of V. If $V \in \mathcal{M}_{A,\chi}(G)$, then V is a vector space over $\widetilde{\mathbb{F}}$ and thus V and $V_{A,\chi}$ are naturally isomorphic. The skew coinvariants define a functor

$$\mathcal{S} : \mathcal{M}(G) \longrightarrow \mathcal{M}_{A,\chi}(G),\ V \mapsto V_{A,\chi},\ \varphi \mapsto \varphi_{A,\chi} = 1 \otimes \varphi,$$

which is left adjoint to the inclusion functor $\mathcal{M}_{A,\chi}(G) \hookrightarrow \mathcal{M}(G)$. The equivalence of categories from Theorem 4.5.2 implies that we get a corresponding skew invariants functor $\mathcal{M}(\mathcal{H}) \longrightarrow \mathcal{M}_{A,\chi}(\mathcal{H})$, left adjoint to the inclusion functor $\mathcal{M}_{A,\chi}(\mathcal{H}) \longrightarrow \mathcal{M}(\mathcal{H})$, where $\mathcal{M}_A(\mathcal{H})$ and $\mathcal{M}_{A,\chi}(\mathcal{H})$ are as defined in Section 4.5.

Proposition 5.1.1. *The category $\mathcal{M}_{A,\chi}(\mathcal{H})$ has enough projectives.*

Proof. Let $N \in \mathcal{M}_{A,\chi}(\mathcal{H})$, $n \in N$. Since $\mathcal{M}_{A,\chi}(\mathcal{H}) \subseteq \mathcal{M}(\mathcal{H})$, there exists a compact open subgroup $H \leqslant G$, such that $H * n = n$. Thus, we can define a map $\varphi : \mathcal{H}\Lambda_H \to N$ by $\varphi(\Theta\Lambda_H) = \Theta * n$. The image of the corresponding map $\varphi_{A,\chi} : (\mathcal{H}\Lambda_H)_{A,\chi} \to N$ contains n.

It remains to show that $(\mathcal{H}\Lambda_H)_{A,\chi}$ is projective. The module $\mathcal{H}\Lambda_H$ is projective in $\mathcal{M}(\mathcal{H})$ [46, I.5.2]. Hence, $(\mathcal{H}\Lambda_H)_{A,\chi}$ is projective in $\mathcal{M}_{A,\chi}(\mathcal{H})$: the inclusion functor is right exact, and since coinvariants is left adjoint to it, it takes projective objects to projective objects. $\square$

Recall from Section 4.2 that for a closed subgroup $H \leqslant G$, we have the induction functor Ind_H^G, which is right adjoint to the restriction functor Res_H^G, and also the compact induction functor $c - \mathrm{Ind}_H^G$, which is left adjoint to Res_H^G, and is defined if $H \leqslant G$ is also open.

Lemma 5.1.2. *[32] (cf. [55, I.5.9]) Let G be a locally compact totally disconnected group. Suppose $H \geqslant A$ is a subgroup of G, closed and compact modulo A. Then Ind_H^G takes injective objects to injective objects. If H is open then $c - \mathrm{Ind}_H^G$ takes projective objects to projective objects.*

Moreover, if the field $\mathbb{F}$ is H/A-ordinary and $(\sigma, W) \in \mathcal{M}_A(H)$, then $\mathrm{Ind}_H^G(\sigma)$ is an injective object and $c - \mathrm{Ind}_H^G(\sigma)$ is a projective object, as soon as H is open.

Proof. Frobenius reciprocity tells us that Ind_H^G is right adjoint to Res_H^G. Any right adjoint to a left exact functor takes injective objects to injective objects. Similarly, since H is open, again by Frobenius reciprocity, $c - \mathrm{Ind}_H^G$ is left adjoint to the restriction functor Res_H^G, which is exact. Any such functor takes projective objects to projective objects.

In the case when $\mathbb{F}$ is H/A-ordinary, $\mathcal{M}_A(H)$ is semisimple by Lemma 4.3.9, and so every $(\sigma, W) \in \mathcal{M}_A(H)$ is a semisimple H-module. In other words, W is both injective and projective. We are done by the first part. $\square$

Note that in the lemma above we do not use the Hecke algebra of G. So we do not need to additionally assume that there exists a compact subgroup $K \leqslant G$, such that $\mathbb{F}$ is K-ordinary. The ordinarity assumption is purely for the purposes that it is a necessary condition for $\mathcal{M}_A(H)$ to be a semisimple category (Lemma 4.3.9).

Corollary 5.1.3. *Let G be a locally compact totally disconnected group. Suppose $H \geqslant A$ is a subgroup of G, open and compact modulo A. Further suppose that the field $\mathbb{F}$ is H/A-ordinary. If (σ, W) is a representation in $\mathcal{M}_A(H)$, then $\mathbb{F}G \underset{\mathbb{F}H}{\otimes} W$ is a projective object in $\mathcal{M}_A(G)$. The statement is also true if we replace $\mathcal{M}_A(H)$ with $\mathcal{M}_{A,\chi}(H)$.*

Proof. Follows from Lemma 5.1.2 and the fact that for H is open, $\mathbb{F}G \underset{\mathbb{F}H}{\otimes} W = a - \mathrm{Ind}_H^G \cong c - \mathrm{Ind}_H^G$ ($a - \mathrm{Ind}_H^G$ as defined in Section 4.2). $\square$

Note that if A is trivial and $\mathbb{F}$ is H-ordinary, for some compact open subgroup $H \leqslant G$, Corollary 5.1.3 tell us that smooth representations of G algebraically induced from a compact open subgroup are projective.

Lemma 5.1.4. *Let G be a locally compact totally disconnected group. Suppose $\mathbb{F}$ is the trivial representation of G and*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{F} \to 0$$

is a projective resolution of $\mathbb{F}$ in $\mathcal{M}_A(G)$. Let $(\pi, V) \in \mathcal{M}_A(G)$, with $\dim_{\mathbb{F}} V$ *arbi-*

trary. Then

$$0 \to P_n \otimes V \to P_{n-1} \otimes V \to \cdots \to P_0 \otimes V \to V \to 0$$

is a projective resolution for V in $\mathcal{M}_A(G)$. The statement is also true if we replace $\mathcal{M}_A(G)$ with $\mathcal{M}_{A,\chi}(G)$.

Proof. We will prove the statement for $\mathcal{M}_A(G)$, but the proof is the same for $\mathcal{M}_{A,\chi}(G)$. For the result to hold, it is enough to show that $P_i \otimes V$ is a projective object in $\mathcal{M}_A(G)$ for all $i = 1, .., n$.

First we observe that $\mathrm{Hom}_{\mathcal{M}_A(G)}(P_i \otimes V, \_\_) \cong \mathrm{Hom}_{\mathcal{M}_A(G)}(P_i, \mathrm{Hom}_{\mathbb{F}}(V, \_\_))$. Let $(\sigma, W) \in \mathcal{M}_A(G)$, then we have maps:

$$\Phi : \mathrm{Hom}_{\mathcal{M}_A(G)}(P_i \otimes V, W) \to \mathrm{Hom}_{\mathcal{M}_A(G)}(P_i, \mathrm{Hom}_{\mathbb{F}}(V, W))$$

$$(\alpha : p_i \otimes v \to w_i) \mapsto (\widetilde{\alpha} : p_i \to (f_\alpha : v \mapsto w_i = \alpha(p_i \otimes v)))$$

and

$$\Psi : \mathrm{Hom}_{\mathcal{M}_A(G)}(P_i, \mathrm{Hom}_{\mathbb{F}}(V, W)) \to \mathrm{Hom}_{\mathcal{M}_A(G)}(P_i \otimes V, W)$$

$$(\beta : p_i \mapsto (\beta_{p_i} : V \to W)) \mapsto (\widetilde{\beta} : p_i \otimes v \mapsto \beta_{p_i}(v)).$$

It is easy to check that Φ and Ψ are inverse to each other. Now, since P_i is projective, the functor $\mathrm{Hom}_{\mathcal{M}_A(G)}(P_i, \_\_)$ is exact. As V is a free $\mathbb{F}$-module $\mathrm{Hom}_{\mathbb{F}}(V, \_\_)$ is also exact. The composition of two exact functors is exact, so $\mathrm{Hom}_{\mathcal{M}_A(G)}(P_i \otimes V, \_\_)$ is exact and $P_i \otimes V$ is projective. $\square$

5.2 Actions on simplicial sets

Before we move on to defining an appropriate notion of an action of the locally compact totally disconnected group G on a simplicial set $\mathcal{X}_\bullet$, we would like to properly introduce simplicial sets and to establish notation which we would use later on. We do so by following Gelfand and Manin [27, Ch. I].

We start by defining the abstract n-simplex $\mathring{\Delta}_n$. It is a topological space

$$\mathring{\Delta}_n = \{(\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}_{>0} | \sum_{k=0}^{n} \alpha_k = 1\}.$$

One defines the i-th vertex e_i of $\mathring{\Delta}_n$ as the point

$$e_i = (0, \dots, 0, 1, 0, \dots),$$

where 1 is in the i-th entry. Note that the vertices of $\mathring{\Delta}_n$ are ordered: $e_0 < e_1 < ... < e_n$. Now let $[m]$ denote the set $\{1, \dots, m\}$. For every non-decreasing map of sets $f : [m] \to [n]$ one associates the m-the face of $\mathring{\Delta}_n$ which is given by the unique linear mapping $\mathring{\Delta}_f : \mathring{\Delta}_n \to \mathring{\Delta}_m$ that preserves the order of the vertices. With this in mind we make the following definition:

Defintion 5.2.1. [27, I.2.1] A *simplicial set* is a family of sets $\mathcal{X}_\bullet = (\mathcal{X}_n)$, for $n = 0, 1, ...,$ and of maps $\mathcal{X}(f) : \mathcal{X}_n \to \mathcal{X}_m$, one for each non-decreasing map $f : [m] \to [n]$, with $[n] := \{0, 1, ..., n\}$, $n \in \mathbb{N}$, such that the following hold:

- $\mathcal{X}(\mathrm{id}) = \mathrm{id}$,
- $\mathcal{X}(g \circ f) = \mathcal{X}(f) \circ \mathcal{X}(g)$.

The maps $\mathcal{X}(f)$ are called *face maps*, and the elements of $\mathcal{X}_n$ *n-simplices*.

For any non-decreasing map $f : [m] \to [n]$, we define the *f-th face* $\mathring{\Delta}_f$ *of* $\mathcal{X}_n$ as the linear map $\mathring{\Delta}_m \to \mathring{\Delta}_n$, that maps any vertex $e_i \in \mathring{\Delta}_m$ to $e_{f(i)} \in \mathring{\Delta}_n$, where $\mathring{\Delta}_n$ is the abstract n-simplex and $i = 0, ..., m$.

Defintion 5.2.2. [27, I.2.9] Let $\mathcal{X}_\bullet$ be a simplicial set and $x \in \mathcal{X}_n$ an n-simplex. Then x is said to be *degenerate* if and only if there exists a surjective non-decreasing map $f : [n] \to [m]$, where $m \leqslant n$, and an element $y \in \mathcal{X}_m$, such that $x = \mathcal{X}(f)(y)$.

Let $\mathcal{X}_{(n)}$ denote the set of non-degenerate n-simplices of $\mathcal{X}_\bullet$.

Defintion 5.2.3. (cf. [27, I.2.2]) The geometric realisation $|\mathcal{X}|$ of a simplicial set $\mathcal{X}_\bullet$ is the topological space with the underlying set $\coprod_n \mathring{\Delta}_n \times \mathcal{X}_{(n)}/R$, where R is the weakest equivalence relation identifying points $(s, x) \in \mathring{\Delta}_n \times \mathcal{X}_n$, and $(t, y) \in \mathring{\Delta}_m \times \mathcal{X}_m$, if

$$y = \mathcal{X}(f)(x), \text{ and } s = \mathring{\Delta}_f(t),$$

for some non-decreasing map $f : [m] \to [n]$. The topology on $|\mathcal{X}|$ is such that quotienting by R is continuous.

We obtain a canonical bijection ([27, I.2.9, Proposition 10])

$$\mathring{\tau} : \coprod_n \mathring{\Delta}_n \times \mathcal{X}_{(n)} \to |\mathcal{X}|. \tag{5.1}$$

We are ready to define an action of a locally compact totally disconnected group G on a simplicial set $\mathcal{X}_\bullet$.

Defintion 5.2.4. We say that G *acts* on the simplicial set $\mathcal{X}_\bullet$ if G acts continuously (i.e., with open stabilisers) on each discrete set $\mathcal{X}_n$ and the action *respects* the face maps $\mathcal{X}(f)$.

We can write this action by

$$\mathbf{g} \cdot \big((\alpha_i), x\big) = \big(F(\mathbf{g}, x)(\alpha_i), \mathbf{g} \cdot x\big) \tag{5.2}$$

where $F(\mathbf{g}, x)$ is an auto-homeomorphism of the abstract n-simplex.

Since G acts continuously on each discrete n-simplex $\mathcal{X}_n$, then the stabilisers in G of all simplices are open. It is worth noting that the respect of the face maps does not necessarily mean that the action commutes with the face maps $\mathcal{X}(f)$. Recall the standard notation [27]:

- $\partial^i = \partial^i_n : [n-1] \to [n]$ is the unique increasing map, missing the value i, i.e., $\partial^i : 1 \mapsto 1, \ldots, i-1 \mapsto i-1, i \mapsto i+1, \ldots, n-1 \mapsto n$,
- $\sigma^i = \sigma^i_n : [n+1] \to [n]$ is the unique non-decreasing surjective map, assuming the value i twice, i.e., $\sigma^i : 1 \mapsto 1, i \mapsto i, i+1 \mapsto i, i+2 \mapsto i+1, \ldots, n+1 \mapsto n$.

Thus, we have the basic face and degeneration maps $\mathcal{X}(\partial^i_n) : \mathcal{X}_n \to \mathcal{X}_{n-1}$ and $\mathcal{X}(\sigma^i_n) : \mathcal{X}_n \to \mathcal{X}_{n+1}$. In particular, given $x \in \mathcal{X}_n$, its codimension one faces are

$\mathcal{X}(\partial_n^i)(x)$ for various i. The codimension one faces of $\mathbf{g}\cdot x$ are $\mathbf{g}\cdot\mathcal{X}(\partial^i)(x)$ but their order could be different. Let us define the maps

$$R = R_n : G\times\mathcal{X}_n \to S_{n+1} = \mathrm{Sym}([n]) \quad \text{by} \quad \mathbf{g}\cdot\mathcal{X}(\partial^i)(x) = \mathcal{X}(\partial^{R(\mathbf{g},x)(i)})(\mathbf{g}\cdot x). \tag{5.3}$$

We want an algebraic criterion on R, which would give us a continuous action of G on $|\mathcal{X}|$. To obtain this we would need the *symmetric crossed simplicial group* $\mathbb{S}_\bullet$ [23]. Recall that $\mathbb{S}_\bullet$ is a simplicial set with $\mathbb{S}_n = S_{n+1}$ and the face maps generated by

$$\mathbb{S}(\partial_n^i)(\phi) = \sigma_{n-1}^i \circ \phi \circ \partial_n^{\phi^{-1}(i)}, \;\; \mathbb{S}(\sigma_n^i)(\phi)(k) = \begin{cases} i, & \text{if } i = \phi(k), \\ i+1, & \text{if } i = \phi(k-1), \\ (\sigma_n^i)^{-1}\phi\,\sigma_n^{\phi^{-1}(i)}(k) & \text{otherwise,} \end{cases}$$

where $\phi \in \mathbb{S}_n$ and $k \in [n]$. We are ready for our definition:

Defintion 5.2.5. Let G be a locally compact totally disconnected group, $\mathcal{X}_\bullet$ a simplicial set and $R = R_n : G\times\mathcal{X}_n \to S_{n+1}$ maps as defined above. We call $(G\times\mathcal{X}, R)$ a *crossed simplicial groupoid*, if the following hold:

$$R(1,x) = 1,\;\; R(\mathbf{gh},x) = R(\mathbf{g},\mathbf{h}\cdot x)R(\mathbf{h},x), \quad \text{for all}\;\; \mathbf{g},\mathbf{h}\in G,\; x\in\mathcal{X}_n, \tag{$\maltese$1}$$

$$\mathbb{S}(f)(R_n(\mathbf{g},x)) = R_m(\mathbf{g},\mathcal{X}(f)(x)), \quad \text{for all}\;\; f:[m]\to[n],\; \mathbf{g}\in G,\; x\in\mathcal{X}_n, \tag{$\maltese$2}$$

$$\mathbf{g}\cdot\mathcal{X}(\partial_n^i)(x) = \mathcal{X}(\partial_n^{R(\mathbf{g},x)(i)})(\mathbf{g}\cdot x), \quad \text{for all}\;\; \mathbf{g}\in G,\; x\in\mathcal{X}_n,\; i\in[n], \tag{$\maltese$3}$$

$$\mathbf{g}\cdot\mathcal{X}(\sigma_n^i)(x) = \mathcal{X}(\sigma_n^{R(\mathbf{g},x)(i)})(\mathbf{g}\cdot x), \quad \text{for all}\;\; \mathbf{g}\in G,\; x\in\mathcal{X}_n,\; i\in[n]. \tag{$\maltese$4}$$

So the maps R are both simplicial and groupoid maps that also compute the permutations of codimension one faces, as well as degenerations, in line with Equation (5.3). Notice that it suffices to verify Condition ($\maltese$2) only for $f = \partial_n^i$ and $f = \sigma_n^i$ for all i and n. Additionally, if the maps R_n are independent in the second argument of $x \in \mathcal{X}_n$, then all these conditions are equivalent to saying that

G, turned to the trivial simplicial group $\mathbb{G}_\bullet$ with $\mathbb{G}_n = G$, $\mathbb{G}(f) = \mathrm{Id}_G$, is a crossed simplicial group $\mathbb{G}$ [23].

Being a crossed simplicial groupoid is sufficient to obtain a continuous action of a group G on $|\mathcal{X}|$. We have the following:

Proposition 5.2.6. *(cf. [23, Prop 1.7]) Let $\mathcal{X}_\bullet$ be a simplicial set with an abstract group G acting on each $\mathcal{X}_n$. Given a system of functions $R = (R_n)$, where $R_n : G \times \mathcal{X}_n \to S_{n+1}$, the following two statements are equivalent:*

1. - *The group G acts on the topological space $|\mathcal{X}|$ by the following simplification of Formula (5.2):*

$$\mathbf{g} \cdot \big((\alpha_i), x\big) = \big((\alpha_{R(\mathbf{g},x)(i)}), \mathbf{g} \cdot x\big),$$

   - *the G-action on degenerate simplices agree with Condition (✚4),*
   - *the maps R are given by Formula (5.3).*

2. *The maps R satisfy Conditions (✚1), (✚2), (✚3) and (✚4).*

Example 5.2.7. Let $G = \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field. Then G has a Bruhat-Tits building $\mathcal{BT}$, which is a simplicial complex of dimension $n = \mathrm{rank}(G)$ [5] (c.f. Section 7.1 for more details on $\mathcal{BT}$). We can think of $\mathcal{BT}$ as a simplicial set $\mathcal{BT}_\bullet = (\mathcal{BT}_k)$, where $\mathcal{BT}_k$ is the set of k-simplices of $\mathcal{BT}$. The group G has a generalised (B,N)-pair structure (I,N), where I is its Iwahori subgroup and N is the subgroup of monomial matrices. Let us compute the generalised Weyl group $\widetilde{W}$ of G. Recall that $\widetilde{W} = W \rtimes \Omega$, where W is the standard Weyl group and Ω is the complementary subgroup (for more details on generalised (B,N)-pairs look at Section 7.4.1). We have

$$I \cap N = \mathrm{Diag}_n(\mathcal{O}_{\mathbb{K}}^\times, \ldots, \mathcal{O}_{\mathbb{K}}^\times) \cong (\mathcal{O}_{\mathbb{K}}^\times)^n,$$

which consists of diagonal matrices with coefficients in the ring of integers $\mathcal{O}_{\mathbb{K}} \leqslant \mathbb{K}$. Denote $T = \mathrm{Diag}_n(\mathbb{K}^\times, \ldots, \mathbb{K}^\times) \cong (\mathbb{K}^\times)^n$ and $H = I \cap N$. Then $N/(I \cap N) \cong$

$N/T \ltimes T/H \cong \mathrm{S}_n \ltimes \mathbb{Z}^n$ is the generalised Weyl group of G, also called the affine Weyl group. It contains the standard Weyl group $W = \mathrm{S}_n \ltimes \mathbb{Z}_0^n$ of type $\widetilde{A}_{n-1}$ as a normal subgroup (where $\mathbb{Z}_0^n = \{(x_i) \mid \sum_i x_i = 0\}$) and we have the complementary group $\Omega = \langle(1, 0, \ldots, 0) \cdot \gamma\rangle$, where $\gamma = (1, 2, \ldots, n) \in S_n$. Now, G acts on $\mathcal{BT}_\bullet$ continuously: stabilisers of all simplices are open. The action also extends to the geometric realisation $|\mathcal{BT}|$: the maps R_n come from the action of Ω on $\mathcal{BT}_n$. More precisely, the generalised Weyl group acts as follows: W acts by preservation of labels, and Ω acts on each n-simplex by permuting its vertices. Thus, to obtain the maps R_n we look at the image of an element $\mathbf{g} \in G$ in $\widetilde{W}$.

5.3 The projective dimension of $\mathcal{M}(G)$

Let $\mathcal{C}$ be a category with enough projectives. Then for every $V \in Ob(\mathcal{C})$ we can construct a resolution

$$\mathscr{P}: \quad \ldots \to P_n \to P_{n-1} \to \ldots \to P_0 \to V \to 0,$$

where $P_i \in Ob(\mathcal{C})$ are projective objects. We say that the resolution is of *length* n if it is of the form

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to V \to 0.$$

The *projective dimension* of an object V is the smallest n, such that

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to V \to 0$$

is a projective resolution of V in $\mathcal{C}$. In this case, we write $\operatorname{proj.dim}(V) = n$. The *projective dimension of the category* $\mathcal{C}$ is defined as

$$\operatorname{proj.dim}(\mathcal{C}) := \sup_{V \in Ob(\mathcal{C})} \operatorname{proj.dim}(V).$$

We are finally ready for the main theorem of this chapter, whose idea goes back to Bernstein (cf. [4, IV.4.2]).

Theorem 5.3.1. *Let G be a locally compact totally disconnected group, A its closed central subgroup and $K \leqslant G$ a compact subgroup, such that the field $\mathbb{F}$ is K-ordinary and thus $\mathcal{H}(G, \mathbb{F}, \mu_K)$ is defined. Suppose G acts continuously on an n-dimensional simplicial set $\mathcal{X}_\bullet$ with contractible geometric realisation $|\mathcal{X}|$, so that A acts trivially on $\mathcal{X}_\bullet$. Suppose that the action of G extends to $|\mathcal{X}|$ (as in Proposition 5.2.6). Suppose further that the stabiliser G_x of any non-degenerate simplex $x \in \mathcal{X}_{(k)}$ is not only open (which follows from continuity) but also compact modulo A. If the field $\mathbb{F}$ is additionally G_x/A-ordinary for any $x \in \mathcal{X}_k$, then*

$$\operatorname{proj.dim}(\mathcal{M}_{A,\chi}(G)) \leqslant n \quad \textit{and} \quad \operatorname{proj.dim}(\mathcal{M}_A(G)) \leqslant n.$$

Proof. Note that the result holds for $\mathbb{F}$, such that $\operatorname{char}(\mathbb{F}) = 0$. Thus, our assumptions on ordinarity are only necessary if one wants to work in positive characteristic. Now, since $\mathbb{F}$ is K-ordinary, then the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of G as defined in Section 4.4 exists. Thus, one can apply Proposition 5.1.1 and Theorem 4.5.3 to deduce that the categories $\mathcal{M}_{A,\chi}(G)$ and $\mathcal{M}_A(G)$ have enough projectives. Thus, projective resolutions of all objects in these categories exist and so we can talk about their projective dimension.

Now let $\mathcal{M}_{A,1}(G)$ denote the category $\mathcal{M}_{A,\chi}(G)$, where χ is the trivial character. The simplicial homology complex of $\mathcal{X}$

$$d_k : C^\sharp_k(\mathcal{X}_\bullet, \mathbb{F}) \to C^\sharp_{k-1}(\mathcal{X}_\bullet, \mathbb{F}),\ d_k\Big(\sum_{x \in \mathcal{X}_k} \alpha_x x\Big) := \sum_{x \in \mathcal{X}_k} \sum_{i=0}^{k} (-1)^i \alpha_x\, \mathcal{X}(\partial^i_k)(x) \tag{5.4}$$

is a complex of smooth G-modules in $\mathcal{M}_{A,1}(G)$ under

$$\mathbf{g} \cdot \Big(\sum_{x \in \mathcal{X}_k} \alpha_x x\Big) := \sum_{x \in \mathcal{X}_k} (-1)^{sign(R(\mathbf{g},x))} \alpha_{\mathbf{g}\cdot x}\, (\mathbf{g} \cdot x). \tag{5.5}$$

If $x = \mathcal{X}(\sigma^i)(y)$ then $y = \mathcal{X}(\partial^i)(x) = \mathcal{X}(\partial^{i+1})(x)$ while all the other faces $\mathcal{X}(\partial^j)(x)$ are degenerate. Hence, $d_k(x)$ is a linear combination of degenerate simplices and the spans of degenerate simplices form a subcomplex of submodules $(C^\flat_k(\mathcal{X}_\bullet, \mathbb{F}), d_k)$.

Let $X_k = C_k(\mathcal{X}_\bullet, \mathbb{F}) := C^\sharp_k(\mathcal{X}_\bullet, \mathbb{F})/C^\flat_k(\mathcal{X}_\bullet, \mathbb{F})$. The $\mathbb{F}$-vector space X_k has a basis $[x]$ with various non-degenerate simplices $x \in \mathcal{X}_{(k)}$. It is still a smooth G-module in $\mathcal{M}_{A,1}(G)$. The spaces X_k comprise the chain complex

$$\mathscr{C}: \quad X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} X_0,$$

that computes the homology of $|\mathcal{X}|$. Since $|\mathcal{X}|$ is contractible, all homology groups are trivial except $H_0(\mathscr{C}) \cong \mathbb{F}$. This yields the exact sequence:

$$0 \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} X_0 \to \mathbb{F} \to 0. \tag{5.6}$$

Let $\mathbb{F}[x]$ be the span of $[x]$ for $x \in \mathcal{X}_{(k)}$. The stabiliser G_x acts on $\mathbb{F}[x]$ by

$$\rho : G_x \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[x]), \rho(\mathbf{g}) = (-1)^{sign(R(\mathbf{g},x))}.$$

Since A acts trivially on $\mathcal{X}_\bullet$, it also acts trivially on $\mathbb{F}[x]$, so $(\rho, \mathbb{F}[x]) \in \mathcal{M}_{A,1}(G)$. Since G_x is open and compact modulo A and $\mathbb{F}$ is G_x/A-ordinary, we can apply Corollary 5.1.3 to deduce that $\mathbb{F}G \bigotimes_{\mathbb{F}G_x} \mathbb{F}[x]$ is a projective object in $\mathcal{M}_{A,1}(G)$.

Let $\mathcal{X}_{(k)}(G)$ be a complete set of representatives of G-orbits on $\mathcal{X}_{(k)}$. We have a G-module isomorphism

$$\sum_{x \in \mathcal{X}_{(k)}(G)} \mathbb{F}G \bigotimes_{\mathbb{F}G_x} \mathbb{F}[x] \xrightarrow{\cong} X_k, \quad \mathbf{g} \otimes \alpha[x] \mapsto \alpha[\mathbf{g} \cdot x], \text{ for } \mathbf{g} \in G, \alpha \in \mathbb{F}.$$

However, as

$$\sum_{x \in \mathcal{X}_{(k)}(G)} \mathbb{F}G \bigotimes_{\mathbb{F}G_x} \mathbb{F}[x]$$

is a sum of projective objects, it is also projective. Thus, the sequence (5.6) is a projective resolution of $\mathbb{F}$ in $\mathcal{M}_{A,1}(G)$.

Let $(\pi, V) \in \mathcal{M}_{A,\chi}(G)$. By Lemma 5.1.4

$$0 \to X_n \otimes V \to X_{n-1} \otimes V \to \cdots \to X_0 \otimes V \to V \to 0$$

is a projective resolution of V of length at most n. This concludes the proof for

$\mathcal{M}_{A,\chi}(G)$. Since $\mathcal{M}_A(G) = \oplus_\chi \mathcal{M}_{A,\chi}(G)$, we get $\operatorname{proj.dim}(\mathcal{M}_A(G)) \leqslant n$. $\square$

Example 5.3.2. Let $G = \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field and $Z(G) = \mathbb{K}^\times$ is the centre of G. Let π be a uniformizer in $\mathbb{K}$. Set $A = \langle \pi^n \rangle$ as our closed central subgroup. The group G has a Bruhat-Tits building $\mathcal{BT}$ which is a simplicial complex of dimension equal to the rank of G. One can see the Bruhat-Tits building as a simplicial set: $\mathcal{BT}_\bullet = (\mathcal{BT}_k)$, where $\mathcal{BT}_k$ is the set of k-simplices of $\mathcal{BT}$. It is a classical result of Bruhat and Tits that $\mathcal{BT}_\bullet$, and thus $|\mathcal{BT}|$, is contractible. Moreover, the group G acts on $\mathcal{BT}_\bullet$ with compact open stabilisers. Therefore, all conditions of Theorem 5.3.1 are satisfied and we obtain that $\operatorname{proj.dim}(\mathcal{M}_A(\mathrm{GL}_n(\mathbb{K}))) \leqslant n$. Notice that our result is subtler than the one observed by Bernstein [4, Th. 29]: he considers groups that act on their Bruhat-Tits buildings with a trivial kernel. However, $\mathrm{GL}_n(\mathbb{K})$ acts with a non-trivial kernel, namely, $Z(G)$. Thus, to apply Bernstein's Theorem we need to quotient by $Z(G)$ and so we only obtain the bound $\operatorname{proj.dim}(\mathcal{M}(\mathrm{PGL}_n(\mathbb{K}))) \leqslant n$, where $(\mathcal{M}(\mathrm{PGL}_n(\mathbb{K}))$ is the category of smooth representations of $\mathrm{PGL}_n(\mathbb{K})$ with no central subgroup involved.

Chapter 6

Cosheaves, localisations and more projective resolutions

To the best of my knowledge the material presented in this chapter is original. It is based on a joint paper by Dmitriy Rumynin and the author of this thesis [32]. If a known result is used it is appropriately referenced. The main references for the non-original material are [27], [52], [25]. Throughout the chapter G denotes a locally compact totally disconnected group, $A \leqslant G$ is a closed central subgroup, $K \leqslant G$ is a compact subgroup, such that the field $\mathbb{F}$ is K-ordinary, and finally $\mathcal{X}_\bullet$ is a simplicial set.

6.1 Cosheaves

In the previous chapter, we defined simplicial sets, their geometric realisation and an action of a locally compact totally disconnected group on both. Now we turn our attention to the next construction involving simplicial sets - sheaves and cosheaves on them. We start by defining a sheaf and a cosheaf on $\mathcal{X}_\bullet$. All our sheaves and cosheaves will have coefficients in $\mathbb{F}$-vector spaces. Since we are not using the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of G in this section, we do not impose any restrictions on the characteristic of $\mathbb{F}$.

Defintion 6.1.1. (cf. [27, I.7.8]) A *cosheaf* $\mathcal{C}$ on $\mathcal{X}_\bullet = (\mathcal{X}_n)$ is a family of $\mathbb{F}$-vector spaces $\{\mathcal{C}_x\}$, one for every simplex $x \in \mathcal{X}_n$ for the various n, and a family of $\mathbb{F}$-linear maps $\mathcal{C}(f, x) : \mathcal{C}_x \to \mathcal{C}_{\mathcal{X}(f)x}$, for each pair (x, f), where $x \in \mathcal{X}_n$, and non-decreasing $f : [m] \to [n]$, such that the following conditions hold:

(C1) $\mathcal{C}(\mathrm{id}, x) = \mathrm{id}$ for every x,

(C2) $\mathcal{C}(g \circ f, x) = \mathcal{C}(g, \mathcal{X}(f)x) \circ \mathcal{C}(f, x)$.

Similarly, a *sheaf* $\mathcal{F}$ on $\mathcal{X}_\bullet$ is a family of $\mathbb{F}$-vector spaces $\{\mathcal{F}_x\}$, one for each $x \in \mathcal{X}_n$, and a family of $\mathbb{F}$-linear maps $\mathcal{F}(f, x) : \mathcal{F}_{\mathcal{X}(f)x} \to \mathcal{F}_x$, for each pair (x, f), where $x \in \mathcal{X}_n$, and non-decreasing $f : [m] \to [n]$, subject to:

(S1) $\mathcal{F}(\mathrm{id}, x) = \mathrm{id}$ for every x,

(S2) $\mathcal{F}(g \circ f, x) = \mathcal{F}(f, x) \circ \mathcal{F}(g, \mathcal{X}(f)x)$.

Note that we have a change in terminology to Gelfand and Manin: we call a homological coefficient system on $\mathcal{X}_\bullet$ *a cosheaf* and a cohomological coefficient system - *a sheaf*. The main motivation for this change lies in the fact that a sheaf $\mathcal{F}$ on $\mathcal{X}_\bullet$ determines a constructible sheaf $|\mathcal{F}|$ on the geometric realisation $|\mathcal{X}|$. The canonical bijection (5.1) permits an explicit description of the stalk $|\mathcal{F}|$ at a point $p \in |\mathcal{X}|$:

$$|\mathcal{F}|_{(p)} = \mathcal{F}_x \ \text{ where } \ \mathring{\tau}(\alpha, x) = p,$$

while the restrictions are determined by the linear structure maps $\mathcal{F}(f, x) : \mathcal{F}_{\mathcal{X}(f)x} \to \mathcal{F}_x$, where $f : [m] \to [n]$. Similarly, a cosheaf $\mathcal{C}$ gives rise to a constructible cosheaf $|\mathcal{C}|$ on $|\mathcal{X}|$.

Again we fix a locally compact totally disconnected topological group G which acts continuously on $\mathcal{X}_\bullet$ and $|\mathcal{X}|$, with $A \leqslant G$ a closed central subgroup, which acts trivially. We will be interested in the following sheaves and cosheaves on $\mathcal{X}_\bullet$, which behave “nicely” with respect to the G-action:

Defintion 6.1.2. An *equivariant cosheaf* is a cosheaf $\mathcal{C}$ with additional data: a linear map $\mathbf{g}_x = \mathbf{g}(\mathcal{C})_x : \mathcal{C}_x \to \mathcal{C}_{\mathbf{g}x}$ for any $\mathbf{g} \in G$ and any simplex x. This data satisfies three axioms:

(i) $\mathbf{g}_{\mathbf{h}x} \circ \mathbf{h}_x = (\mathbf{gh})_x$ for any $\mathbf{g}, \mathbf{h} \in G$ and a simplex x.

(ii) $\mathcal{C}_x$ is a smooth representation of G_x for any simplex x, where G_x denotes the stabiliser of x in G.

(iii) The square
$$\begin{tikzcd}
\mathcal{C}_x \arrow[r, "\mathbf{g}_x"'] \arrow[d, "\mathcal{C}(f,x)"] & \mathcal{C}_{\mathbf{g}x} \arrow[d, "\mathcal{C}(\mathbf{g}f,\mathbf{g}x)"] \\
\mathcal{C}_{\mathcal{X}(f)x} \arrow[r, "\mathbf{g}_{\mathcal{X}(f)x}"] & \mathcal{C}_{\mathcal{X}(\mathbf{g}f)\mathbf{g}x}
\end{tikzcd}$$
is commutative for all $\mathbf{g} \in G$, simplices $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

Defintion 6.1.3. A *morphism* $\psi : \mathcal{C} \to \mathcal{D}$ of equivariant cosheaves is a system of linear maps $\psi_x : \mathcal{C}_x \to \mathcal{D}_x$, commuting with actions and corestrictions, i.e, the squares

$$\begin{tikzcd}
\mathcal{C}_x \arrow[r, "\psi_x"'] \arrow[d, "\mathcal{C}(f,x)"] & \mathcal{D}_x \arrow[d, "\mathcal{D}(f,x)"] \\
\mathcal{C}_{\mathcal{X}(f)x} \arrow[r, "\psi_{\mathcal{X}(f)x}"] & \mathcal{D}_{\mathcal{X}(f)x}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C}_x \arrow[r, "\psi_x"'] \arrow[d, "\mathbf{g}(\mathcal{C})_x"] & \mathcal{D}_x \arrow[d, "\mathbf{g}(\mathcal{D})_x"] \\
\mathcal{C}_{\mathbf{g}x} \arrow[r, "\psi_{\mathbf{g}x}"] & \mathcal{D}_{\mathbf{g}x}
\end{tikzcd}$$

are commutative for all $\mathbf{g} \in G$, $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

Defintion 6.1.4. An *equivariant sheaf* is a sheaf $\mathcal{F}$ with additional data: a linear map $\mathbf{g}_x = \mathbf{g}(\mathcal{F})_x : \mathcal{F}_x \to \mathcal{F}_{\mathbf{g}x}$ for any $\mathbf{g} \in G$ and any simplex x. This data satisfies three axioms:

(i) $\mathbf{g}_{\mathbf{h}x} \circ \mathbf{h}_x = (\mathbf{gh})_x$ for any $\mathbf{g}, \mathbf{h} \in G$ and a simplex x.

(ii) $\mathcal{F}_x$ is a smooth representation of G_x for any simplex x.

(iii) The square
$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\mathbf{g}_x"'] & \mathcal{F}_{\mathbf{g}x} \\
\mathcal{F}_{\mathcal{X}(f)x} \arrow[r, "\mathbf{g}_{\mathcal{X}(f)x}"] \arrow[u, "\mathcal{F}(f,x)"'] & \mathcal{F}_{\mathcal{X}(\mathbf{g}f)\mathbf{g}x} \arrow[u, "\mathcal{F}(\mathbf{g}f,\mathbf{g}x)"']
\end{tikzcd}$$
is commutative for all $\mathbf{g} \in G$, simplices $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

Defintion 6.1.5. A *morphism* $\psi : \mathcal{F} \to \mathcal{E}$ of equivariant sheaves is a system of linear maps $\psi_x : \mathcal{F}_x \to \mathcal{E}_x$, commuting with actions and restrictions, i.e, the squares

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow[\psi_x]{} & \mathcal{E}_x \\ \uparrow{\scriptstyle \mathcal{F}(f,x)} & & \uparrow{\scriptstyle \mathcal{E}(f,x)} \\ \mathcal{F}_{\mathcal{X}(f)x} & \xrightarrow{\psi_{\mathcal{X}(f)x}} & \mathcal{E}_{\mathcal{X}(f)x} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}_x & \xrightarrow[\psi_x]{} & \mathcal{E}_x \\ \downarrow{\scriptstyle \mathbf{g}(\mathcal{F})_x} & & \downarrow{\scriptstyle \mathbf{g}(\mathcal{E})_x} \\ \mathcal{F}_{\mathbf{g}x} & \xrightarrow{\psi_{\mathbf{g}x}} & \mathcal{E}_{\mathbf{g}x} \end{array}$$

are commutative for all $\mathbf{g} \in G$, $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

We denote the category of equivariant cosheaves by $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ and by $\mathrm{Sh}_G(\mathcal{X}_\bullet)$ the category of equivariant sheaves. They are both abelian categories - the kernels and cokernels can be computed simplex-wise [52, V].

By definition of an equivariant cosheaf $\mathcal{C}$ on $\mathcal{X}_\bullet$, $\mathcal{C}_x$ is a smooth representation of the simplex stabiliser G_x. Thus, we have a G_x-action on $\mathcal{C}_x$ given by a map $\pi_x : G \to \mathrm{Aut}_{\mathbb{F}}(\mathcal{C}_x)$. Let $K_x = \ker(\pi_x)$. Clearly, the G_x-action factors through the quotient G_x/K_x. Note that since the action of G on the simplicial set $\mathcal{X}_\bullet$ is continuous, then every G_x is open in G. Thus, if K_x is open in G_x, then K_x is open in G. Let $\dot{\mathbf{g}} \in G_x/K_x$. Then $\dot{\mathbf{g}} = \mathbf{g}K_x$, for some $\mathbf{g} \in G_x$. Therefore, $\dot{\mathbf{g}}$ is open as a homeomorphic image of an open subgroup. In particular, G_x/K_x is discrete. All of this remains true if we replace a cosheaf $\mathcal{C}$ by a sheaf $\mathcal{F}$, and $\mathcal{C}_x$ by $\mathcal{F}_x$. In both cases we say that the action of G_x on $\mathcal{C}_x$ (or on $\mathcal{F}_x$) *factors through a discrete quotient.*

Motivated by this discussion we make the following definition:

Defintion 6.1.6. An equivariant cosheaf $\mathcal{C}$ (sheaf $\mathcal{F}$) on $\mathcal{X}_\bullet$ is called *discrete*, if the stabiliser G_x of any simplex x of $\mathcal{X}_\bullet$ acts on $\mathcal{C}_x$ (or $\mathcal{F}_x$) through a discrete quotient.

Denote by $\mathrm{Csh}^\circ_G(\mathcal{X}_\bullet)$ the full subcategory of $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ of discrete equivariant cosheaves and by $\mathrm{Sh}^\circ_G(\mathcal{X}_\bullet)$ the full subcategory of $\mathrm{Sh}_G(\mathcal{X}_\bullet)$ of discrete equivariant sheaves. These categories are abelian [52, V.].

Note that since the closed central subgroup A acts trivially, $A \leqslant G_x$, for every simplex x. Thus, we can define a full subcategory of $\mathrm{Sh}_G(\mathcal{X}_\bullet)$ ($\mathrm{Csh}_G(\mathcal{X}_\bullet)$) consisting of of A-semisimple (co)sheaves, i.e., those (co)sheaves where each $\mathcal{F}_x$ (correspondingly $\mathcal{C}_x$) is an A-semisimple smooth representation of G_x.

Furthermore, we have a version of A-semisimple (co)sheaves with a fixed character χ. In this case, for every simplex x, $\mathcal{F}_x$ (correspondingly $\mathcal{C}_x$) are objects of $\mathcal{M}_{A,\chi}(G_x)$. Hence, we have six categories of equivariant cosheaves (and similarly sheaves):

$$\begin{tikzcd}
\mathrm{Csh}_G(\mathcal{X}_\bullet) & \mathrm{Csh}_{G,A}(\mathcal{X}_\bullet) \arrow[l, "\supseteq"'] & \mathrm{Csh}_{G,A,\chi}(\mathcal{X}_\bullet) \arrow[l, "\supseteq"'] \\
\mathrm{Csh}^\circ_G(\mathcal{X}_\bullet) \arrow[u, "\subseteq"'] & \mathrm{Csh}^\circ_{G,A}(\mathcal{X}_\bullet) \arrow[l, "\supseteq"] \arrow[u, "\subseteq"'] & \mathrm{Csh}^\circ_{G,A,\chi}(\mathcal{X}_\bullet). \arrow[l, "\supseteq"] \arrow[u, "\subseteq"']
\end{tikzcd}$$

Similarly to the standard setting of sheaves and cosheaves, we also have a notion of a *trivial sheaf* and a *trivial cosheaf.*

Defintion 6.1.7. With conventions and notation as before, let $(\pi, V) \in \mathcal{M}(G)$. The *trivial cosheaf* $\underset{\sim}{V}$ on $\mathcal{X}_\bullet$ and *the trivial sheaf* $\underline{\underline{V}}$ are defined as follows:

$$\underline{\underline{V}}_x = \underset{\sim}{V}_x := V, \quad \underset{\sim}{V}(f,x) := \mathrm{Id}_V, \quad \underline{\underline{V}}(f,x) := \mathrm{Id}_V, \quad \mathbf{g}_x := \pi(\mathbf{g}),$$

for all $\mathbf{g} \in G$, $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

Note that the trivial cosheaf $\underset{\sim}{V}$ is discrete if and only if the trivial sheaf $\underline{\underline{V}}$ is discrete, which is true if and only if G_x acts on V through a discrete quotient, for every simplex x. Similarly, $\underset{\sim}{V}$ is A-semisimple if and only if $\underline{\underline{V}}$ is A-semisimple, if and only if $V \in \mathcal{M}_A(G)$.

We would like to construct more interesting and fruitful examples of sheaves and cosheaves on $\mathcal{X}_\bullet$. Therefore, we introduce the following notion:

Defintion 6.1.8. Let G be a locally compact totally disconnected group acting on a simplicial set $\mathcal{X}_\bullet$. A *system of subgroups* $\mathcal{G}$ of G is a datum assigning a subgroup $\mathcal{G}_x$ of the simplex stabiliser G_x to each simplex $x \in \mathcal{X}_n$. The datum needs to be G-equivariant, i.e., $\mathbf{g}\mathcal{G}_x\mathbf{g}^{-1} = \mathcal{G}_{\mathbf{g}x}$ for all $\mathbf{g} \in G$ and $x \in \mathcal{X}_n$. We call a system of subgroups $\mathcal{G}$:

- *open* if $\mathcal{G}_x$ is open in G_x for all x.
- *closed* if $\mathcal{G}_x$ is closed in G_x for all x.
- *cofinite* if the index of $\mathcal{G}_x$ in G_x is finite for all x.

- *compact modulo A* if $\mathcal{G}_x$ is compact modulo A for all x.
- *contravariant* if $\mathcal{G}_{\mathcal{X}(f)x} \subseteq \mathcal{G}_x$ for all $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.
- *covariant* if $\mathcal{G}_{\mathcal{X}(f)x} \supseteq \mathcal{G}_x$ for all $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$.

Note that the G-equivariance condition implies that $\mathcal{G}_x$ is a normal subgroup of G_x for any simplex x.

Example 6.1.9. (cf. [52]) Let $G = \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field with ring of integers $\mathcal{O}_{\mathbb{K}}$ and π a uniformizer in $\mathbb{K}$. Let $\mathcal{BT}$ be the Bruhat-Tits building of G. The vertices of $\mathcal{BT}$ are equivalence classes $[\mathcal{L}]$ of $\mathcal{O}_{\mathbb{K}}$-lattices in $\mathbb{K}^n$, where two lattices $\mathcal{L}, \mathcal{L}'$ are equivalent if $\mathcal{L}' = \lambda\mathcal{L}$, for some $\lambda \in \mathbb{K}^\times$. The k-simplices are families $\{[\mathcal{L}_0], \ldots, [\mathcal{L}_k]\}$, where $\mathcal{L}_i, i = 0, \ldots, k$ satisfy the condition

$$\mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq \ldots \subsetneq \pi^{-1}\mathcal{L}_0.$$

For more details on $\mathcal{BT}$ see Section 7.1. As we discussed in Chapter 5 we can think of $\mathcal{BT}$ as a simplicial set $\mathcal{BT}_\bullet = (\mathcal{BT}_k)$, where $\mathcal{BT}_k$ is the set of k-simplices. We give an example of an open, compact, contravariant system of subgroups of G on $\mathcal{BT}_\bullet$. Fix $n \in \mathbb{N}_{>0}$. Let $x_0 \in \mathcal{BT}_0$ denote the vertex of $\mathcal{BT}$ given by the equivalence class of the lattice $\mathcal{O}^n$. Define the subgroup $\mathcal{G}_{x_0} \leqslant G_{x_0}$ as

$$\mathcal{G}_{x_0} \coloneqq \{\mathbf{g} \in \mathrm{GL}_n(\mathcal{O}) \mid \mathbf{g} \equiv 1 \bmod \pi^n\}.$$

This subgroup is compact open. Now for any other vertex $x \in \mathcal{BT}_0$ there exists an element $\mathbf{g} \in G$, such that $x = \mathbf{g}x_0$. Thus, we define

$$\mathcal{G}_x \coloneqq \mathbf{g}\mathcal{G}_{x_0}\mathbf{g}^{-1}.$$

For any k-simplex $y \in \mathcal{BT}_k$ with vertices $x_1, \ldots, x_{k+1}$ let

$$\mathcal{G}_y = \langle \mathcal{G}_{x_1} \cup \ldots \cup \mathcal{G}_{x_{k+1}} \rangle.$$

This is clearly a contravariant system of subgroups of G (by construction) and it is compact and open since each $\mathcal{G}_y$ is, for $y \in \mathcal{BT}_k$, $k = 0, \ldots, n$.

Recall that for a smooth representation $(\pi, V) \in \mathcal{M}(G)$ we have the space of *invariants*

$$V^G := \{v \in V \mid \pi(\mathbf{g})v = v, \text{ for all } \mathbf{g} \in G\}$$

and the space of *coinvariants*

$$V_G := V/V', \text{ where } V' = \langle \{\pi(\mathbf{g})v - v\}_{v \in V, \mathbf{g} \in G} \rangle_{\mathbb{F}}.$$

Observe the following:

Lemma 6.1.10. *If $\mathcal{G}$ is a system of subgroups of G and $(\pi, V) \in \mathcal{M}(G)$, then for every simplex x, the invariants $V^{\mathcal{G}_x}$ and coinvariants $V_{\mathcal{G}_x}$ are smooth representations of the simplex stabiliser G_x.*

Proof. Let x be any simplex and $G_x \leqslant G$ its stabiliser. Let $\pi : G \to \mathrm{Aut}_{\mathbb{F}}(V)$. We want to show that π restricts to a map

$$\bar{\pi} : G_x \to \mathrm{Aut}_{\mathbb{F}}(V^{\mathcal{G}_x}), \quad \mathbf{g} \mapsto \pi(\mathbf{g}).$$

Let $\mathbf{g} \in G_x$. Then

$$\mathbf{g} \cdot V^{\mathcal{G}_x} = \pi(\mathbf{g})V^{\mathcal{G}_x} = \{\pi(g)v \mid v \in V^{\mathcal{G}_x}\} \subseteq V^{\mathbf{g}\mathcal{G}_x\mathbf{g}^{-1}} = V^{\mathcal{G}_{\mathbf{g}x}} = V^{\mathcal{G}_x}.$$

The last two equalities hold by equivariance of $\mathcal{G}$ and the fact that $\mathbf{g} \in G_x$. Thus, $\bar{\pi}$ is defined. Let $v \in V^{\mathcal{G}_x} \subseteq V$. Then $\mathrm{Stab}_{G_x}(v) = \mathrm{Stab}_G(v) \cap G_x$. Since $(\pi, V) \in \mathcal{M}(G)$, $\mathrm{Stab}_G(v)$ is open in G and thus $\mathrm{Stab}_{G_x}(v)$ is open in G_x in the subspace topology, finishing the proof for invariants. For coinvariants, let

$$\pi|_{G_x} : G_x \to \mathrm{Aut}_{\mathbb{F}}(V)$$

be the restriction of π to G_x. Since $(\pi, V) \in \mathcal{M}(G)$, then $(\pi|_{G_x}, V) \in \mathcal{M}(G_x)$. It follows that $V_{\mathcal{G}_x}$ is also a smooth representation of G_x as any quotient of a smooth representation is a smooth representation [9]. $\square$

Lemma 6.1.10 enables us to construct G-equivariant sheaves and cosheaves on $\mathcal{X}_\bullet$ in the following way:

Proposition 6.1.11. *Let $\mathcal{G}$ be a system of subgroups of G and $(\pi, V) \in \mathcal{M}(G)$. The following statements hold:*

1. *If $\mathcal{G}$ is contravariant, then the invariants $\underset{\approx}{V^{\mathcal{G}}}_x := V^{\mathcal{G}_x}$ is an equivariant cosheaf and the coinvariants $\underline{\underline{V_{\mathcal{G}}}}_x := V_{\mathcal{G}_x}$ is an equivariant sheaf.*
2. *If $\mathcal{G}$ is covariant, then the invariants $\underline{\underline{V^{\mathcal{G}}}}_x := V^{\mathcal{G}_x}$ is an equivariant sheaf and the coinvariants $\underset{\approx}{V_{\mathcal{G}}}_x := V_{\mathcal{G}_x}$ is an equivariant cosheaf.*
3. *If, further to (1) or (2), $\mathcal{G}$ is open, then the (co)sheaf is discrete.*
4. *If, further to (1) or (2), V is A-semisimple (with a fixed character χ), then the sheaves $\underline{\underline{V^{\mathcal{G}}}}$, $\underline{\underline{V_{\mathcal{G}}}}$ and the cosheaves $\underset{\approx}{V^{\mathcal{G}}}$, $\underset{\approx}{V_{\mathcal{G}}}$ are A-semisimple (with a fixed character χ correspondingly).*

Proof. Suppose first that $\mathcal{G}$ is contravariant. This means that $\mathcal{G}_{\mathcal{X}(f)x} \subseteq \mathcal{G}_x$ for all $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$. Thus, $V^{\mathcal{G}_x} \subseteq V^{\mathcal{G}_{\mathcal{X}(f)x}}$. Therefore, we obtain an inclusion:

$$\underset{\approx}{V^{\mathcal{G}}}(f, x) : V^{\mathcal{G}_x} \to V^{\mathcal{G}_{\mathcal{X}(f)x}}.$$

This is clearly an $\mathbb{F}$-linear map. As it is just an inclusion, then properties (C1) and (C2) of Definition 6.1.1 are trivially satisfied. For every $\mathbf{g} \in G$, define a map

$$\mathbf{g}_x : V^{\mathcal{G}_x} \to V^{\mathcal{G}_{\mathbf{g}x}}, \quad v \mapsto \pi(\mathbf{g})(v).$$

Since π is a representation, $\pi(\mathbf{g})$ is an $\mathbb{F}$-linear map by definition. Also, as π is a homomorphism,

$$\mathbf{g}_{\mathbf{h}_x} \circ \mathbf{h}_x = \pi(\mathbf{g})(\pi(\mathbf{h})(v)) = \pi(\mathbf{gh})(v) = (\mathbf{gh})_x.$$

By Lemma 6.1.10, $V^{\mathcal{G}_x}$ is a smooth representation of G_x, for every simplex x. By the definition of our maps, Property (iii) of Definition 6.1.2 trivially holds. Thus, $\underset{\sim}{V^{\mathcal{G}}}{}_x$ is a G-equivariant cosheaf on $\mathcal{X}_\bullet$.

Now, let us look at coinvariants. Since $\mathcal{G}_{\mathcal{X}(f)x} \subseteq \mathcal{G}_x$ for all $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$, we have a natural surjection:

$$\underline{\underline{V_{\mathcal{G}}}}(f, x) : V_{\mathcal{G}_{\mathcal{X}(f)x}} \to V_{\mathcal{G}_x}.$$

This is an $\mathbb{F}$-linear map, which clearly satisfies conditions (S1) and (S2) of Definition 6.1.1. For the G-equivariant part, for every $\mathbf{g} \in G$, we define the maps

$$\mathbf{g}_x : V_{\mathcal{G}_x} \to V_{\mathcal{G}_{\mathbf{g}x}}, \quad v \mapsto \pi(\mathbf{g})v.$$

These are linear as (π, V) is a representation of G. By Lemma 6.1.10 we know that $(\pi|_{G_x}, V_{\mathcal{G}_x}) \in \mathcal{M}(G_x)$. The final condition of Definition 6.1.4 is trivially satisfied. Thus, $\underline{\underline{V_{\mathcal{G}}}}{}_x := V_{\mathcal{G}_x}$ is a G-equivariant sheaf on $\mathcal{X}_\bullet$. This finishes the proof of (1).

The proof of (2) is analogous to (1). The only difference is that for a covariant system $\mathcal{G}$, we have inclusions $\mathcal{G}_x \subseteq \mathcal{G}_{\mathcal{X}(f)x}$ for all $x \in \mathcal{X}_n$ and non-decreasing maps $f : [m] \to [n]$, giving rise to maps

$$\underset{\sim}{V^{\mathcal{G}}}(f, x) : V^{\mathcal{G}_{\mathcal{X}(f)x}} \to V^{\mathcal{G}_x},$$

which define a G-equivariant sheaf. We also have natural surjections

$$\underline{\underline{V_{\mathcal{G}}}}(f, x) : V_{\mathcal{G}_x} \to V_{\mathcal{G}_{\mathcal{X}(f)x}},$$

which, together with linear maps $\mathbf{g}_x$ as in (1), give rise to a G-equivariant cosheaf.

By definition, $\mathcal{G}_x$ acts trivially on both $V^{\mathcal{G}_x}$ and $V_{\mathcal{G}_x}$, for every simplex x. Thus, the G_x-action on $V^{\mathcal{G}_x}$ and $V_{\mathcal{G}_x}$ factors through $G_x/\mathcal{G}_x$, which is discrete if $\mathcal{G}$ is open. This finishes the proof of (3).

Suppose $(\pi, V) \in \mathcal{M}_A(G)$ (or $\mathcal{M}_{A,\chi}(G)$). Since a submodule and a quotient of a semisimple module is semisimple, if follows that $V_{\mathcal{G}_x}, V^{\mathcal{G}_x} \in \mathcal{M}_A(G)$ (or $\mathcal{M}_{A,\chi}(G)$)

for every simplex x, which proves (4). $\square$

Example 6.1.12. Recall that for $G = \mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field and π is a uniformizer in $\mathbb{K}$, in Example 6.1.9 we constructed an open, compact, cotravariant system of subgroups of G. Let $A = \langle \pi^n \rangle \leqslant Z(G)$ be a closed central subgroup of G and let $(\pi, V) \in \mathcal{M}_A(G)$. We have a G-equivariant cosheaf $\underset{\sim}{V^{\mathcal{G}}}_x = (V^{\mathcal{G}_x})$ on $\mathcal{BT}_\bullet$ for a k-simplex $y \in \mathcal{BT}_k$, $k = 0, \ldots, n$, given by

$$V^{\mathcal{G}_y} = \{v \in V \mid \pi(\mathbf{g})v = v \text{ for all } \mathbf{g} \in \mathcal{G}_y\}.$$

Cosheaves turn out to be more fruitful than sheaves for studying representations in our setting. So from now on we turn our attention to cosheaves. Recall our notation from Section 5.3:

$$C_n^\sharp(\mathcal{X}_\bullet, \mathcal{C}) := \Big\{ \sum_{x \in \mathcal{X}_n} \alpha_x x \;\Big|\; \alpha_x \in \mathcal{C}_x, \text{ all but finitely many } \alpha_x = 0 \Big\},$$

$$d_0 := 0, \quad d_n\Big(\sum_{x \in \mathcal{X}_n} \alpha_x x \Big) := \sum_{x \in \mathcal{X}_n} \sum_{i=0}^{n} (-1)^i [\mathcal{C}(\partial_n^i, x)(\alpha_x)][\mathcal{X}(\partial_n^i)(x)]$$

for $n > 0$. Since degenerate simplices span a subcomplex $(C_n^\flat(\mathcal{X}_\bullet, \mathcal{C}), d_n)$, we are mainly interested in the quotient complex

$$C_k(\mathcal{X}_\bullet, \mathcal{C}) := C_k^\sharp(\mathcal{X}_\bullet, \mathcal{C}) / C_k^\flat(\mathcal{X}_\bullet, \mathcal{C})$$

which is spanned by linear combinations of non-degenerate simplices $\sum_{x \in \mathcal{X}_{(n)}} \alpha_x[x]$. Before we proceed to the next proposition, which establishes basic properties of the chain groups defined above, we want to introduce some further subcategories of the category of smooth representations $\mathcal{M}(G)$. Let $U \leqslant G$ be an open subgroup. Consider the category $\mathcal{M}(G)^U \subseteq \mathcal{M}(G)$ with objects those smooth representations, that are generated by their U-fixed vectors. This is a full subcategory of $\mathcal{M}(G)$. Let $\mathcal{M}(G)^\circ$ be the union of various $\mathcal{M}(G)^U$. Its objects are those smooth representations that are generated by U-fixed vectors for some open subgroup $U \subseteq G$. Inside them we have the corresponding A-semisimple categories

$$\mathcal{M}_A(G)^U,\ \mathcal{M}_A(G)^\circ,\ \mathcal{M}_{A,\chi}(G)^U \ \text{ and } \ \mathcal{M}_{A,\chi}(G)^\circ.$$

The chain groups $C_n(\mathcal{X}_\bullet, \mathcal{C})$ satisfy the following interesting properties.

Proposition 6.1.13. *Let $\mathcal{C}$ be a G-equivariant cosheaf on $\mathcal{X}_\bullet$. Let $x_1, x_2 \ldots$ be representatives of the G-orbits on $\mathcal{X}_{(n)}$. Then the following statements hold:*

1. *The groups $C_n(\mathcal{X}_\bullet, \mathcal{C})$ and homologies $H_n(\mathcal{X}_\bullet, \mathcal{C})$ are smooth representations of G, for all n.*

2. *There is an isomorphism of G-modules*

$$C_n(\mathcal{X}_\bullet, \mathcal{C}) \cong \bigoplus_k a - \mathrm{Ind}_{G_{x_k}}^G \mathcal{C}_{x_k}.$$

3. *If $\mathcal{C}$ is A-semisimple (with a character χ), then chains $C_n(\mathcal{X}_\bullet, \mathcal{C})$ and homologies $H_n(\mathcal{X}_\bullet, \mathcal{C})$ are A-semisimple (with a character χ respectively).*

4. *If $\mathcal{C}$ is discrete and $\mathcal{X}_{(n)}$ has finitely many G-orbits, then chains $C_n(\mathcal{X}_\bullet, \mathcal{C})$ and homologies $H_n(\mathcal{X}_\bullet, \mathcal{C})$ are in $\mathcal{M}(G)^\circ$. More precisely, $C_n(\mathcal{X}_\bullet, \mathcal{C})$ are in $\mathcal{M}(G)^U$, where $U = U_1 \cap U_2 \cap \ldots \cap U_k$ and U_i is the kernel of the $\mathcal{C}_{x_i}$ representation of G_{x_i}.*

5. *If $\mathcal{X}_{(n)}$ has finitely many G-orbits and $\mathcal{C}_{x_k}$ is finitely generated G_{x_k}-module for each x_k, then chains $C_n(\mathcal{X}_\bullet, \mathcal{C})$ and homologies $H_n(\mathcal{X}_\bullet, \mathcal{C})$ are finitely generated G-modules.*

6. *Suppose that for each $x \in \mathcal{X}_n$, the stabiliser G_x is compact modulo A and the field $\mathbb{F}$ is G_x/A-ordinary. If $\mathcal{C}$ is A-semisimple (with a character χ), then the space of chains $C_n(\mathcal{X}_\bullet, \mathcal{C})$ is a projective object in $\mathcal{M}_A(G)$ (correspondingly in $\mathcal{M}_{A,\chi}(G)$).*

Proof. The G-action on the chains is defined as follows:

$$\mathbf{g} \cdot \Big(\sum_{x \in \mathcal{X}_{(n)}} \alpha_x[x] \Big) := \sum_{x \in \mathcal{X}_{(n)}} (-1)^{sign(R(\mathbf{g},x))} \mathbf{g}_x(\alpha_x) \left[\mathbf{g} \cdot x\right].$$

We need to show that $\mathrm{Stab}_G(c)$ is open, for every $c \in C_n(\mathcal{X}_\bullet, \mathcal{C})$. Write

$$c = \sum_{\substack{x_i \in \mathcal{X}_{(n)}, \\ i=1}}^{m} \alpha_{x_i}[x_i].$$

Clearly, $\mathrm{Stab}_G(c) \subseteq \bigcap_{i=1}^m \mathrm{Stab}_G(x_i)$. However, for $\mathbf{g} \in \bigcap_{i=1}^n \mathrm{Stab}_G(x_i)$, $\mathcal{C}_{\mathbf{g}x_i} = \mathcal{C}_{x_i}$, and so $\mathbf{g}_{x_i} : \mathcal{C}_{x_i} \to \mathcal{C}_{x_i}$ has to be the identity by the definition of a cosheaf. Thus, $\mathrm{Stab}_G(c) = \bigcap_{i=1}^n \mathrm{Stab}_G(x_i)$. By the continuity of the G-action on $\mathcal{X}_\bullet$, each $\mathrm{Stab}_G(x_i)$ is open and, thus, so is $\mathrm{Stab}_G(c)$ as a finite intersection of open sets. Condition (iii) in Definition 6.1.2 implies that the action of G commutes with the chain maps d_n, for all n. Thus, $G \cdot \ker(d_n) \subseteq \ker(d_n)$ and so $\ker(d_n)$ is a smooth representation of G, for every n. This shows that $H_n(\mathcal{X}_\bullet, \mathcal{C})$ is a smooth representation of G for all n, as a quotient of smooth representation is smooth.

Recall that $a - \mathrm{Ind}_{G_{x_k}}^G \mathcal{C}_{x_k} = \mathbb{F}G \otimes_{\mathbb{F}G_{x_k}} \mathcal{C}_{x_k}$. The G-module isomorphism in (2) is given by

$$\mathbf{g}(c_{x_k})[\mathbf{g} \cdot x_k] \longleftrightarrow \mathbf{g} \otimes c_{x_k}.$$

Extend by linearity to obtain the result.

If $\mathcal{C}$ is A-semisimple, then $\mathcal{C}_k \in \mathcal{M}_A(G)$ for every simplex $x_k \in \mathcal{X}_n$. As the action is continuous and G_{x_k} are open in G for every simplex x_k, by Lemma 4.2.1 we have a functor

$$a - \mathrm{Ind}_{G_{x_k}}^G : \mathcal{M}_A(G_{x_k}) \to \mathcal{M}_A(G).$$

Utilising the isomorphism from Statement (2), the result follows since a direct sum of smooth representations is smooth and A-semisimplicity is clearly preserved. The proof is the same if we replace $\mathcal{M}_A(G)$ with $\mathcal{M}_{A,\chi}(G)$.

Elements of $C_n(\mathcal{X}_\bullet, \mathcal{C})$ are finite linear combinations of $\alpha_x x$, where $x \in \mathcal{X}_n$ and $\alpha_x \in \mathcal{C}_x$. Since the action of G on $\mathcal{X}_n$ has finitely many orbits, for every $x \in \mathcal{X}_n$, there exists a $\mathbf{g} \in G$, such that $x = \mathbf{g} \cdot x_i$, for some i. Thus, we can write

$$\alpha_x x = \mathbf{g}_{x_i}(\alpha_{x_i})[\mathbf{g} \cdot x_i] = \mathbf{g} \cdot (\alpha_{x_i} x_i).$$

As U fixes everything, we are done. To prove (5), analogously to (4), we can write every $\alpha_x x$ as

$$\alpha_x x = \mathbf{g}_{x_i}(\alpha_{x_i})[\mathbf{g} \cdot x_i].$$

As all $\mathcal{C}_{x_k}$ are finitely generated G-modules, we can further rewrite $\mathbf{g}_{x_i}(\alpha_{x_i})$ as a finite G-linear combination of elements of $\mathcal{C}_{x_k}$ finishing the proof. Statement (6) follows from (2) and Corollary 5.1.3, where we replace H with G_x. $\square$

6.2 Localisation

In this section we establish an equivalence between the category of smooth representations $\mathcal{M}(G)$ of a locally compact totally disconnected group G, acting on a simplicial set $\mathcal{X}_\bullet$ and a certain localisation of the category $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ of G-equivariant cosheaves on $\mathcal{X}_\bullet$. In this section we make no assumptions on the characteristic of the field $\mathbb{F}$.

Let us first recall the definition of a localisation of a category, or equivalently, of the category of fractions.

Defintion 6.2.1. [25] Let $\mathcal{A}, \mathcal{B}$ be categories and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a functor. $\mathcal{F}$ is said *to make a morphism invertible*, if for a morphism $f : a \to b$ in $\mathcal{A}$, the morphism $\mathcal{F}(f) : \mathcal{F}(a) \to \mathcal{F}(b)$ is invertible in $\mathcal{B}$ (note that by invertible we mean an isomorphism).

Defintion 6.2.2. [25] Let $\mathcal{A}$ be a category and let $Mor(\mathcal{A})$ denote the morphisms in $\mathcal{A}$. Let $\Sigma \subseteq Mor(\mathcal{A})$. To $\mathcal{A}$ and Σ, we associate the pair $(\mathcal{Q}_\Sigma, \mathcal{A}[\Sigma^{-1}])$, where $\mathcal{A}[\Sigma^{-1}]$ is a category and $\mathcal{Q}_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ is a functor, such that:

1. $\mathcal{Q}_\Sigma$ makes the morphisms in Σ invertible.
2. If there is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, which makes the morphisms in Σ invertible, there exists a unique functor $\mathcal{G} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$, such that $\mathcal{F} = \mathcal{G} \circ \mathcal{Q}_\Sigma$.

Let us describe the category $\mathcal{A}[\Sigma^{-1}]$. It is obtained from $\mathcal{A}$ by adding the "inverted" morphisms in Σ. More precisely:

- The objects in $\mathcal{A}[\Sigma^{-1}]$ are the same as the objects of $\mathcal{A}$.
- The morphisms in $\mathcal{A}[\Sigma^{-1}]$ are the union of morphisms in $\mathcal{A}$ and morphisms in Σ^{-1}, where

$$\Sigma^{-1} := \{g \mid g = f^{-1}, \text{ for } f \in \Sigma\}.$$

  Composition of morphisms from $\mathcal{A}$ is done as in $\mathcal{A}$, and for morphisms $f \in Mor(\mathcal{A})$ and $\sigma \in \Sigma^{-1}$, $f \circ \sigma$ exists in $\mathcal{A}[\Sigma^{-1}]$ if $r(\sigma) = d(f)$, where d denotes the domain function and and r the range. Similarly, $\sigma \circ f$ exists if $d(\sigma) = r(f)$.

For $\Sigma \subseteq Mor(\mathcal{A})$, the pair $(\mathcal{Q}_\Sigma, \mathcal{A}[\Sigma^{-1}])$ always exists [25].

Defintion 6.2.3. [25, I.2.2] A subset (class) $\Sigma \subseteq Mor(\mathcal{A})$ is said to *admit a calculus of left fractions* if the following conditions are satisfied:

(LF1) All identities of $\mathcal{A}$ belong to Σ.

(LF2) Σ is closed under compositions, i.e., if $f, g \in \Sigma$ and $f \circ g$ is well-defined, $f \circ g \in \Sigma$.

(LF3) Every diagram
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X' & & Y \end{tikzcd}$$
with $s \in \Sigma, u \in Mor(\mathcal{A})$, can be completed to a commutative square:
$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X' \arrow[dr, dashed, "u'"'] & & Y \arrow[dl, dashed, "t"] \\ & Y' & \end{tikzcd}$$
with $t \in \Sigma$ and $u' \in Mor(\mathcal{A})$.

(LF4) If $f, g : X \to Y$ are morphisms in $\mathcal{A}$ and $s : X' \to X$ is a morphism in Σ, such that $f \circ s = g \circ s$, then there exists a morphism $t : Y \to Y'$ in Σ, such that $t \circ f = t \circ g$.

If $\Sigma \subseteq Mor(\mathcal{A})$ satisfies the dual conditions to (LF1)-(LF4), it is said to *admit a calculus of right fractions.*

Subsets which admit calculi of fractions behave nicely under localisation, which is why we are interested in them. In particular, we have the following useful property:

Lemma 6.2.4. *[25, I.3] Let $\mathcal{A}$ be a category, and $\Sigma \subseteq Mor(\mathcal{A})$ which admits a calculus of left fractions. Then the following hold:*

1. *If $\mathcal{A}$ is additive, then so is $\mathcal{A}[\Sigma^{-1}]$.*

2. *If $\mathcal{A}$ has finite colimits, then so does $\mathcal{A}[\Sigma^{-1}]$.*

In particular, if $\mathcal{A}$ is an abelian category, then $\mathcal{A}[\Sigma^{-1}]$ is an additive category with finite colimits. If Σ admits a calculus of right fractions instead, then $\mathcal{Q}_\Sigma$ preserves finite limits.

Now let us turn our attention back to the categories $\mathcal{M}(G)$ of smooth representations of a locally compact totally disconnected group G, and $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ of G-equivariant cosheaves on the simplicial set $\mathcal{X}_\bullet$, on which G acts continuously. We have the following functors:

$$\mathfrak{L} : \mathcal{M}(G) \to \mathrm{Csh}_G(\mathcal{X}_\bullet), \quad \mathfrak{L}((\rho, V)) = \underset{\sim}{V},$$

where $\underset{\sim}{V}$ denotes the trivial cosheaf as in Definition 6.1.7, and

$$\mathfrak{H} : \mathrm{Csh}_G(\mathcal{X}_\bullet) \to \mathcal{M}(G), \quad \mathfrak{H}(\mathcal{C}) = H_0(\mathcal{X}_\bullet, \mathcal{C}).$$

Note that $H_0(\mathcal{X}_\bullet, \mathcal{C}) \in \mathcal{M}(G)$ by Proposition 6.1.13. Set $\Sigma \subset \mathrm{Mor}(\mathrm{Csh}_G(\mathcal{X}_\bullet))$ to be the class of those morphisms f, such that $\mathfrak{H}(f)$ is an isomorphism. Thus, we can define the localised category $\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]$. We have a functor

$$\mathfrak{H}[\Sigma^{-1}] : \mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}] \to \mathcal{M}(G),$$

obtained as $\mathfrak{H}[\Sigma^{-1}] \circ \mathcal{Q}_\Sigma = \mathfrak{H}$.

We are ready for the main theorem of this section, which is a generalisation of a Localisation Theorem by Schneider and Stuhler [52, Theorem V.1]. We follow their strategy in our proof. It is important to notice that no restriction on $\mathbb{F}$ appears in the theorem.

Theorem 6.2.5 (Localisation Theorem)**.** *Consider a continuous action of the locally compact totally disconnected group G on a simplicial set $\mathcal{X}_\bullet$, where the central subgroup A acts trivially.*

1. *The class Σ of morphisms f in $\mathrm{Csh}_G(\mathcal{X}_\bullet)$, such that $\mathfrak{H}(f)$ is an isomorphism admits a calculus of left fractions.*
2. *$\mathfrak{H}[\Sigma^{-1}] : \mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}] \to \mathcal{M}(G)$ is conservative, i.e., a morphism f is an isomorphism if and only if $\mathfrak{H}[\Sigma^{-1}](f)$ is an isomorphism.*
3. *$\mathfrak{H}[\Sigma^{-1}]$ commutes with colimits.*
4. *$\mathfrak{H}[\Sigma^{-1}]$ is faithful, i.e., injective on morphisms.*

If $|\mathcal{X}|$ is connected, then the following three statements hold:

(5) $\mathfrak{H}[\Sigma^{-1}] : \mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}] \to \mathcal{M}(G)$ is an equivalence of categories.

(6) $\mathcal{Q}_\Sigma \circ \mathfrak{L}$ is a quasi-inverse of $\mathfrak{H}[\Sigma^{-1}]$.

(7) These equivalences restrict to equivalences $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma_A^{-1}] \xrightarrow{\cong} \mathcal{M}_A(G)$ and $\mathrm{Csh}_{G,A,\chi}(\mathcal{X}_\bullet)[\Sigma_{A,\chi}^{-1}] \xrightarrow{\cong} \mathcal{M}_{A,\chi}(G)$ where Σ_A and $\Sigma_{A,\chi}$ are intersections of Σ with the corresponding subcategories.

Proof. To prove the first three statements we use the following result of [25]:

Proposition 6.2.6. *[25, I.3.4] Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor commuting with finite direct limits and $\Sigma \subseteq Mor(\mathcal{A})$, such that for every $s \in \Sigma$, $\mathcal{F}(s)$ is an isomorphism. If finite direct limits exist in $\mathcal{A}$, Σ admits a calculus of left fractions. Moreover, the functor $\mathcal{G} : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$, defined by $\mathcal{G} \circ \mathcal{Q}_\Sigma = \mathcal{F}$, is conservative and commutes with finite direct limits.*

Recall that $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ is an abelian category. Also note that a short exact sequence of cosheaves gives rise to a long exact sequence in homology. Consequently, the functor $\mathfrak{H}$ is right exact. Hence, it commutes with finite direct limits (cf. [36, Prop. 3.3.3], the statement proved there is that a left exact functor commutes with finite inverse limits. Apply the opposite categories to dualise it). Thus, taking $\mathfrak{H} = \mathcal{F}$ in Proposition 6.2.6, Statement (1), (2) and (3) follow.

We move on to (4). Suppose $\mathfrak{H}[\Sigma^{-1}](f) = \mathfrak{H}[\Sigma^{-1}](f')$, for two morphisms f and f'. Lemma 6.2.4 implies that cokernels in $\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]$ exist. Thus, to prove that $f = f'$ it suffices to show that $\mathrm{coker}(f - f')$ is an isomorphism. By (3), $\mathfrak{H}[\Sigma^{-1}](\mathrm{coker}(f - f')) = \mathrm{coker}(\mathfrak{H}[\Sigma^{-1}](f) - \mathfrak{H}[\Sigma^{-1}](f')) = \mathrm{coker}(0)$ is an isomorphism. By (2) $\mathrm{coker}(f - f')$ is an isomorphism, which proves (4).

By definition of $C_n(\mathcal{X}_\bullet, \mathbb{F})$ and $C_n(\mathcal{X}_\bullet, \underset{\sim}{V})$, for $V \in \mathcal{M}(G)$, the tensor product $C_k(\mathcal{X}_\bullet, \mathbb{F}) \otimes V$ is naturally isomorphic as a G-representation to $C_k(\mathcal{X}_\bullet, \underset{\sim}{V})$. Since $|\mathcal{X}|$ is connected, we have an exact sequence

$$C_1(\mathcal{X}_\bullet, \mathbb{F}) \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \mathbb{F}) \xrightarrow{w} \mathbb{F} \to 0, \quad \text{where } w : \sum_x \alpha_x x \mapsto \sum_x \alpha_x .$$

By the comment above, tensoring with V produces another exact sequence

$$C_1(\mathcal{X}_\bullet, \underset{\sim}{V}) \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \underset{\sim}{V}) \to V \to 0.$$

Since

$$\mathfrak{H}[\Sigma^{-1}](\mathcal{Q}_\Sigma(\mathfrak{L}(V))) \cong \mathfrak{H}(\mathfrak{L}(V)) = H_0(\mathcal{X}_\bullet, \underset{\sim}{V}) \xrightarrow{\cong} V,$$

we have a natural isomorphism $\mathfrak{H}[\Sigma^{-1}] \circ (\mathcal{Q}_\Sigma \circ \mathfrak{L}) \cong \mathrm{Id}_{\mathcal{M}(G)}$. In the opposite direction, we need a natural transformation

$$\gamma : \mathrm{Id}_{\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]} \to (\mathcal{Q}_\Sigma \circ \mathfrak{L}) \circ \mathfrak{H}[\Sigma^{-1}].$$

We define it in $\mathrm{Csh}_G(\mathcal{X}_\bullet)$ for each cosheaf $\mathcal{C}$ by

$$\gamma(\mathcal{C})_x := \begin{cases} \mathcal{C}_x \ni \alpha \mapsto 0 \in \mathfrak{H}(\mathcal{C}) & \text{if} \quad x \in \mathcal{X}_n,\ n > 0\,, \\ \mathcal{C}_x \ni \alpha \mapsto [\alpha x] \in \mathfrak{H}(\mathcal{C}) & \text{if} \quad x \in \mathcal{X}_0\,. \end{cases}$$

Observe that $\mathfrak{H}(\gamma(\mathcal{C}))$ is an isomorphism by construction. By (2), $\gamma(\mathcal{C})$ is an isomorphism, so γ is a natural isomorphism. This proves (5) and (6).

To attack (7), let us first compare the categories $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma^{-1}]$ and $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma_A^{-1}]$. The former is a full subcategory of $\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]$, while the latter is the localisation of $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)$. They are connected by a natural functor

$$\mathcal{N} : \mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma_A^{-1}] \to \mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma^{-1}],$$

identity on objects and morphisms. Clearly, $\mathcal{N}$ is an equivalence. It remains to observe $\mathfrak{H}(\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma^{-1}]) \subseteq \mathcal{M}_A(G)$ and $\mathcal{Q}_\Sigma(\mathfrak{L}(\mathcal{M}_A(G))) \subseteq \mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma^{-1}]$. Both inclusions are straightforward. $\square$

Let us again add a system of subgroups to the picture. Let $\mathcal{G}$ be a contravariant system of subgroups of G. We have an exact sequence:

$$C_1(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{w} V, \quad w\Big(\sum_x \alpha_x x\Big) = \sum_x \alpha_x, \tag{6.1}$$

where $\underset{\sim}{V}^{\mathcal{G}}$ denotes the cosheaf of invariants of $\mathcal{G}$ as defined in Proposition 6.1.11.

Using it, we can get a version of Theorem 6.2.5 for discrete cosheaves. Let Σ°, Σ_A° and $\Sigma_{A,\chi}^\circ$ be the intersections of Σ with $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)$, $\mathrm{Csh}_{G,A}^\circ(\mathcal{X}_\bullet)$ and $\mathrm{Csh}_{G,A,\chi}^\circ(\mathcal{X}_\bullet)$ correspondingly.

Corollary 6.2.7. *Suppose that $|\mathcal{X}|$ is connected and there are finitely many G-orbits on $\mathcal{X}_0$. Suppose further that for any representation $V \in \mathcal{M}(G)^\circ$ there exists an open contravariant system of subgroups $\mathcal{G}$ of G, such that the following variation of sequence (6.1) is exact:*

$$C_1(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}}) \xrightarrow{w} V \to 0.$$

Then the functor $\mathfrak{H}[\Sigma^{-1}]$ provides equivalences $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)[\Sigma^{\circ\,-1}] \xrightarrow{\cong} \mathcal{M}(G)^\circ$,

$$\mathrm{Csh}_{G,A}^\circ(\mathcal{X}_\bullet)[\Sigma_A^{\circ\,-1}] \xrightarrow{\cong} \mathcal{M}_A(G)^\circ, \quad \textit{and} \quad \mathrm{Csh}_{G,A,\chi}^\circ(\mathcal{X}_\bullet)[\Sigma_{A,\chi}^{\circ\,-1}] \xrightarrow{\cong} \mathcal{M}_{A,\chi}(G)^\circ.$$

Proof. The relation between the categories $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)[\Sigma^{-1}]$ and $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)[\Sigma^{\circ\,-1}]$ is similar to the relation between $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma^{-1}]$ and $\mathrm{Csh}_{G,A}(\mathcal{X}_\bullet)[\Sigma_A^{-1}]$ in the proof of Theorem 6.2.5. Parts (3) and (4) of Proposition 6.1.13 tell us that $\mathfrak{H}[\Sigma^{-1}]$ is a well-defined functor $\mathrm{Csh}_G^\circ(\mathcal{X}_\bullet)[\Sigma^{\circ\,-1}] \to \mathcal{M}(G)^\circ$, as well as a well-defined functor between the A-semisimple categories.

If $V \in \mathcal{M}(G)^\circ$, we pick a system of subgroups $\mathcal{G}$ as in the statement. Then the trivial cosheaf $\mathcal{Q}_\Sigma(\mathfrak{L}(V)) = \underset{\sim}{V}$ is isomorphic to the cosheaf $\underset{\sim}{V^{\mathcal{G}}}$ in $\mathrm{Csh}_G(\mathcal{X}_\bullet)[\Sigma^{-1}]$. As $\mathcal{G}$ is open, the latter cosheaf is discrete because G_x acts on $\underset{\sim}{V^{\mathcal{G}}}_x$ via the discrete quotient $G_x/\mathcal{G}_x$. The A-semisimplicity follows similarly. □

6.3 Projective resolutions revisited: The Schneider-Stuhler resolution

Let G be a locally compact totally disconnected topological group, $A \leqslant G$ a closed central subgroup, $K \leqslant G$ a compact subgroup, such that the field $\mathbb{F}$ is K-ordinary. Thus, we have a Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of G over $\mathbb{F}$. Suppose that G acts continuously on a simplicial set $\mathcal{X}_\bullet$. Recall that if $\mathcal{C}$ is a G-equivariant cosheaf on $\mathcal{X}_\bullet$, then for all n, the chains $\mathcal{C}_n(\mathcal{X}_\bullet, \mathcal{C})$ are smooth representations of G and can be projective objects (Proposition 6.1.13).

Let $\mathcal{G}$ be a contravariant system of subgroups of G. For $(\pi, V) \in \mathcal{M}_A(G)$ recall the cosheaf of invariants $\underset{\sim}{V^{\mathcal{G}}}_x := (V^{\mathcal{G}_x})_{x \in \mathcal{X}_\bullet}$ on $\mathcal{X}_\bullet$ given by

$$V^{\mathcal{G}_x} := \{v \in V \mid \pi(\mathbf{g})v = v, \text{ for all } \mathbf{g} \in \mathcal{G}_x\}.$$

Acknowledging the construction of Schneider and Stuhler for p-adic reductive groups [52], we propose the following definition:

Defintion 6.3.1. Let G and $\mathcal{X}_\bullet$ be as above. For $V \in \mathcal{M}_A(G)$ (or $\mathcal{M}_{A,\chi}(G)$), *we call a finitely generated projective resolution of* V *of the form* $C_\bullet(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}})$ *a Schneider-Stuhler resolution.*

Our next aim is to find the "right" system of subgroups $\mathcal{G}$ of G to obtain our Schneider-Stuhler resolutions.

Denote by $f_i^n : [0] \to [n]$ the function $f_i^n(0) = i$. Suppose we are given a family of compact open subgroups $\mathcal{G}_x$, one for each vertex $x \in \mathcal{X}_0$, such that the following hold:

(E1) $\mathcal{G}_{\mathbf{g}x} = \mathbf{g}\mathcal{G}_x\mathbf{g}^{-1}$, for all $\mathbf{g} \in G$, $x \in \mathcal{X}_0$.

(E2) $\mathcal{G}_x\mathcal{G}_y = \mathcal{G}_y\mathcal{G}_x$ if x and y are adjacent, i.e., $x = \mathcal{X}(f_0^1)(w)$, $y = \mathcal{X}(f_1^1)(w)$ for some $w \in \mathcal{X}_1$.

Condition (E2) allows us to extend this family of subgroups to a compact open contravariant system of subgroups. For an n-simplex x, we define the relevant subgroup $\mathcal{G}_x$ by taking products over vertices:

$$\mathcal{G}_x := \mathcal{G}_{\mathcal{X}(f_0^n)x}\mathcal{G}_{\mathcal{X}(f_1^n)x}\cdots\mathcal{G}_{\mathcal{X}(f_n^n)x} \text{ for all } x \in \mathcal{X}_n.$$

Defintion 6.3.2. Let G be a locally compact totally disconnected group which acts continuously on a simplicial set $\mathcal{X}_\bullet$. A compact open contravariant system of subgroups $\mathcal{G}$ of G, obtained by the construction above from some initial given family of subgroups $\{\mathcal{G}_{x_0}\}$, for each $x_0 \in \mathcal{X}_0$, is called *an exquisite system.*

Note that if $\mathcal{G}$ is an exquisite system and the field $\mathbb{F}$ is $\mathcal{G}_x$-ordinary for each $x \in \mathcal{X}_0$, then it is $\mathcal{G}_x$-ordinary for each $x \in \mathcal{X}_\bullet$.

Meyer and Solleveld use systems of subgroups of reductive p-adic groups to obtain collections of idempotents [43]. More precisely, an exquisite system of subgroups gives rise to a collection of idempotents $\Lambda_x := \Lambda_{\mathcal{G}_x} \in \mathcal{H}(G, \mathbb{F}, \mu_K)$, where $K \leqslant G$ is some compact subgroup, such that $\mathbb{F}$ is K-ordinary (we use the same notation as in Section 4.3). Let $V \in \mathcal{M}(G)$. Recall Theorem 4.5.2 gives an equivalence of categories $\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H})$, where $\mathcal{M}(\mathcal{H})$ is the category of smooth modules over the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of G. In particular, we can think of V as a smooth $\mathcal{H}(G, \mathbb{F}, \mu_K)$-module. Let $*$ denote the module action. The idempotents have a very nice property - they control the spaces of invariants, more precisely, $V^{\mathcal{G}_x} = \Lambda_x * V$. The proof of the following lemma is sketched for Bruhat-Tits buildings of p-adic groups in the case when $\mathrm{char}(\mathbb{F}) = 0$ by Meyer and Solleveld [43, Lemma 2.6]. We generalise it and give a full proof in the case of an arbitrary locally compact totally disconnected group G acting on a simplicial set $\mathcal{X}_\bullet$ and a field $\mathbb{F}$ whose characteristic could be positive, as long as there exists a compact subgroup $K \leqslant G$, such that $\mathbb{F}$ is K-ordinary.

Lemma 6.3.3. *The collection Λ_x, $x \in \mathcal{X}_\bullet$ of idempotents of $\mathcal{H}$ arisen from an exquisite system of subgroups, satisfies the following identities:*

1. $\Lambda_x \star \Lambda_y = \Lambda_y \star \Lambda_x$ *if $x, y \in \mathcal{X}_0$ are adjacent.*
2. $\Lambda_x = \Lambda_{\mathcal{X}(f_0^n)x} \star \Lambda_{\mathcal{X}(f_1^n)x} \star \ldots \star \Lambda_{\mathcal{X}(f_n^n)x}$ *, for all $x \in \mathcal{X}_n$.*
3. $\Lambda_{\mathbf{g}\cdot x} = {}^{\mathbf{g}^{-1}}\Lambda_x{}^{\mathbf{g}}$*, for all $\mathbf{g} \in G$, $\mathbf{x} \in \mathcal{X}_\bullet$.*

Proof. By definition

$$\Lambda_x \star \Lambda_y(\mathbf{g}) = \int_G \Lambda_x(\mathbf{h})\Lambda_y(\mathbf{h}^{-1}\mathbf{g})\mu(d\mathbf{h}). \tag{6.2}$$

The integrand vanishes unless $\mathbf{h} \in \mathcal{G}_x$, $\mathbf{h}^{-1}\mathbf{g} \in \mathcal{G}_y$. Thus $\Lambda_x \star \Lambda_y$ is supported on $\mathcal{G}_x\mathcal{G}_y$. Moreover, $\mathbf{h}^{-1}\mathbf{g} \in \mathcal{G}_y$ translates into $\mathbf{h} \in \mathbf{g}\mathcal{G}_y$ so that (6.2) becomes

$$\int_{\mathcal{G}_x \cap \mathbf{g}\mathcal{G}_y} \Lambda_x(\mathbf{h})\Lambda_y(\mathbf{h}^{-1}\mathbf{g})\mu(d\mathbf{h}) = \frac{\mu(\mathcal{G}_x \cap \mathbf{g}\mathcal{G}_y)}{\mu(\mathcal{G}_x)\mu(\mathcal{G}_y)}. \tag{6.3}$$

Decomposing $\mathbf{g} = \mathbf{h}(\mathbf{h}^{-1}\mathbf{g})$ for some $\mathbf{h} \in \mathcal{G}_x$, $\mathbf{h}^{-1}\mathbf{g} \in \mathcal{G}_y$, (6.3) becomes

$$\frac{\mu(\mathcal{G}_x \cap \mathbf{h}\mathcal{G}_y)}{\mu(\mathcal{G}_x)\mu(\mathcal{G}_y)} = \frac{\mu(\mathbf{h}^{-1}(\mathcal{G}_x \cap \mathbf{h}\mathcal{G}_y))}{\mu(\mathcal{G}_x)\mu(\mathcal{G}_y)} = \frac{\mu(\mathcal{G}_x \cap \mathcal{G}_y)}{\mu(\mathcal{G}_x)\mu(\mathcal{G}_y)} = \frac{1}{|\mathcal{G}_x : (\mathcal{G}_x \cap \mathcal{G}_y)|\mu(\mathcal{G}_y)} =$$

$$\frac{1}{|\mathcal{G}_x\mathcal{G}_y : \mathcal{G}_y|\mu(\mathcal{G}_y)} = \frac{1}{\mu(\mathcal{G}_x\mathcal{G}_y)} = \Lambda_{\mathcal{G}_x\mathcal{G}_y}(\mathbf{g}).$$

Since $\mathcal{G}_x\mathcal{G}_y = \mathcal{G}_y\mathcal{G}_x$, we have proved not only (1) but a stronger equation

$$\Lambda_x \star \Lambda_y = \Lambda_{\mathcal{G}_x\mathcal{G}_y} = \Lambda_y \star \Lambda_x. \tag{6.4}$$

Statement (2) follows from Equation (6.4) by induction. The third statement follows from G-equivariance of $\mathcal{G}$:

$$\Lambda_{\mathbf{g}x} = \Lambda_{\mathcal{G}_{\mathbf{g}x}} = \Lambda_{\mathbf{g}\mathcal{G}_x\mathbf{g}^{-1}} = {}^{\mathbf{g}^{-1}}\Lambda_x{}^{\mathbf{g}}.$$

□

Recall the following standard definitions from topology:

Defintion 6.3.4.
- A topological space X is called a *unique geodesic space*, if there exists a unique geodesic $[x, y]$ between any two points $x, y \in X$.
- A subset $Y \subseteq X$ is called *convex* if $[x, y] \subseteq Y$ for all $x, y \in Y$.
- *The convex hull* $\mathfrak{Hull}(Y)$ *of* Y is the intersection of all convex subsets of X containing Y. Notice that $[x, y] = \mathfrak{Hull}(\{x, y\})$.

Back to our setting, $|\mathcal{X}|$ is the geometric realisation of the simplicial set $\mathcal{X}_\bullet = (\mathcal{X}_n)$. For a non-degenerate $x \in \mathcal{X}_{(n)}$ we denote the corresponding simplex in $|\mathcal{X}|$ by $\mathring{\Delta}_n \times x$ and its points by $\mathbf{x} = (\alpha, x)$, $\mathbf{y} = (\alpha, y)$, etc. Denote by $\mathring{x} = ((\frac{1}{n+1}, \dots \frac{1}{n+1}), x)$ the centre of a simplex x. We would like to make the additional assumption that the space $|\mathcal{X}|$ admits a CAT(0)-metric. The consequence of this which is relevant to our investigation is that a CAT(0)-space is a unique geodesic space. Denote the unique geodesic between any two points $\mathbf{x}, \mathbf{y} \in |\mathcal{X}|$ by $[\mathbf{x}, \mathbf{y}]$.

For a system of subgroups $\mathcal{G}$ of G we would like to have some control over the subgroups $\mathcal{G}_x$, along geodesics. Bearing this in mind, we propose the following definition:

Defintion 6.3.5. We say that a contravariant system of subgroups $\mathcal{G}$ is *geodesic* if for all $\mathbf{x}, \mathbf{y} \in |\mathcal{X}|$

$$\mathcal{G}_z \subseteq \mathcal{G}_x\mathcal{G}_y,$$

where $z \in \mathcal{X}_0$ is a vertex of the first simplex $u \in \mathcal{X}_n$ along the geodesic $[\mathbf{x}, \mathbf{y}]$, i.e., $z = \mathcal{X}(f_i^n)u$ for some i and $(\Delta_n \times u) \cap [\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{v}]$ for some $\mathbf{v} \in]\mathbf{x}, \mathbf{y}]$.

The significance of this definition transpires in the following lemma, inspired by similar results of Meyer and Solleveld for Bruhat-Tits buildings of p-adic groups:

Lemma 6.3.6. *(cf. [43, Prop 2.2 and Lemma 2.6]) Let G be a locally compact totally disconnected group, $K \leqslant G$ a compact subgroup, such that $\mathbb{F}$ is K-ordinary. Suppose that $|\mathcal{X}|$ admits a CAT(0)-metric, $\mathcal{G}$ is a geodesic exquisite system of subgroups of G and the field $\mathbb{F}$ is $\mathcal{G}_x$-ordinary for each $x \in \mathcal{X}_0$. Then*

$$\Lambda_x \star \Lambda_z \star \Lambda_y = \Lambda_x \star \Lambda_y \quad \textit{and} \quad \Lambda_x \star \Lambda_z = \Lambda_z \star \Lambda_x,$$

as soon as $x, y, z \in \mathcal{X}_\bullet$ satisfy the conditions spelled out in Definition 6.3.5.

Proof. If $z = \mathcal{X}(f_i^n)u$ as in Definition 6.3.5, then Λ_x is a product of various $\Lambda_{\mathcal{X}(f_k^n)u}$, hence, commutes with Λ_z. The first equality easily follows from the geodesic condition $\mathcal{G}_z \subseteq \mathcal{G}_x\mathcal{G}_y$. □

Now consider a character $\chi : A \to \widetilde{\mathbb{F}}^\times$. Given a subgroup $H \leqslant G$, set $H_\chi := H/H \cap \ker(\chi)$. It is a subgroup of G_χ. Observe that H_χ is compact if and only if H is compact modulo A. Recall that a smooth representation $(\pi, V) \in \mathcal{M}(G)$ is called *admissible* if the invariants V^U is a finite-dimensional subspace of V for every compact open subgroup $U \leqslant G$.

We would like to state the following conjecture:

Conjecture 6.3.7. Let G be a locally compact totally disconnected group, A its closed central subgroup, $K \leqslant G$ a compact subgroup, such that $\mathbb{F}$ is K-ordinary. Suppose G acts continuously on a simplicial set $\mathcal{X}_\bullet$ of dimension n, with A acting trivially. Further suppose that a face of a non-degenerate simplex in $\mathcal{X}_\bullet$ is non-degenerate and $|\mathcal{X}|$ admits a CAT(0)-metric, such that the faces are geodesic, i.e., $\mathfrak{Hull}(\mathring{\Delta}_n \times x) = \mathring{\Delta}_n \times x$, for each $x \in \mathcal{X}_{(\bullet)}$. If $V \in \mathcal{M}_{A,\chi}(G)$, the following four statements should conjecturally hold:

1. If $\mathcal{G}$ is a geodesic exquisite system of subgroups of G_χ, such that $\mathbb{F}$ is $\mathcal{G}_x$-ordinary for all $x \in \mathcal{X}_0$, then the complex

$$0 \to C_n(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}}) \xrightarrow{d_n} C_{n-1}(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}}) \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}}) \xrightarrow{w} V$$

   is an exact sequence.

2. Each $C_k(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}})$ is a projective module in $\mathcal{M}_{A,\chi}(G)$.

3. If (π, V) is generated by invariants $V^{\mathcal{G}_x}$ for some $x \in \mathcal{X}_0$, then the complex is a projective resolution of V in $\mathcal{M}_{A,\chi}(G)$.

4. If (π, V) is admissible and $\mathcal{X}_{(k)}$ has finitely many G-orbits, then $C_k(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}})$ is a finitely generated G-module.

First note that the assumption that there exists a compact subgroup $K \leqslant G$, such that $\mathbb{F}$ is K-ordinary, implies that there exists a Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_K)$ of G over $\mathbb{F}$ as defined in Section 4.4. Therefore, Theorem 4.5.2, Corollary 4.5.3 and Proposition 5.1.1 imply that $\mathcal{M}_A(G)$ and $\mathcal{M}_{A,\chi}(G)$ have enough projectives. Thus, we can discuss projective resolutions of objects. Secondly, note that admissibility implies that each $V^{\mathcal{G}_x}$ is a finitely generated G-module. Thus, in fact Statements (2)–(4) are established in Proposition 6.1.13. Only Statement (1) is truly a conjecture. It is proved for affine Bruhat-Tits buildings by Meyer and Solleveld [43, Theorem 2.4]. We can prove its partial case:

Theorem 6.3.8. *If the dimension of $|\mathcal{X}|$ is one, then Conjecture 6.3.7 holds.*

Proof. **(1), exactness at $C_0(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}})$:** The inclusion $\mathrm{im}(d_1) \subseteq \ker(w)$ is clear: it follows from the fact that V is generated by invariants $V^{\mathcal{G}_x}$ for some vertex $x \in \mathcal{X}_0$. Let us show that $\ker(w) \subseteq \mathrm{im}(d_1)$. Pick a 0-cycle $\alpha = \sum_{i=1}^{n} \alpha_i x_i \in C_0(\mathcal{X}_\bullet, \underset{\sim}{V^{\mathcal{G}}})$ where all $\alpha_i \neq 0$. Consider the hull of its support $Y := \mathfrak{Hull}(\{\mathring{x}_1, \ldots, \mathring{x}_n\})$. Under our conditions $|\mathcal{X}|$ is a tree, so Y is a finite tree. Hence, Y has an endpoint. Without loss of generality, $\mathring{x}_1$ is an endpoint. Let $x_1' \in \mathcal{X}_0$ be the unique vertex adjacent to x_1, such that $\mathring{x}_1' \in Y$. Let $e_1 \in \mathcal{X}_1$ be the edge connecting x_1 and x_1'. Since $w(\alpha) = \sum_{i=1}^{n} \alpha_i = 0$ and $\Lambda_{x_i}(\alpha_i) = \alpha_i$, we conclude that

$$\sum_{i=1}^{n} \Lambda_{x_i}(\alpha_i) = 0. \tag{6.5}$$

Applying $\Lambda_{x_1} \star (1 - \Lambda_{x_1'})$ to Equation (6.5), we can rewrite each summand separately, using Lemmas 6.3.3 and Lemma 6.3.6:

- $\Lambda_{x_1} \star (1 - \Lambda_{x_1'}) \star \Lambda_{x_1}(\alpha_1) = (1 - \Lambda_{x_1'})(\alpha_1)$,
- $\Lambda_{x_i} \star (1 - \Lambda_{x_1'}) \star \Lambda_{x_i}(\alpha_i) = 0$ for $i \geqslant 2$.

Thus, $\alpha_1 \in \ker(1 - \Lambda_{x_1'})$ and $\alpha_1 \in \operatorname{im}(\Lambda_{x_1'})$. Then

$$\alpha' := \alpha_1 x_1' + \sum_{i=2}^{n} \alpha_i x_i = d_1(\pm\alpha e_1) + \alpha \in C_0(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}})$$

and the hull of the support of α' is a proper subset of Y. An easy induction on the size of the hull of the support completes the proof.

(1), exactness at $C_1(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}})$**:** Pick a 1-cycle $\alpha = \sum_{i=1}^{n} \alpha_i x_i \in C_1(\mathcal{X}_\bullet, \underset{\sim}{V}^{\mathcal{G}})$ where all $\alpha_i \neq 0$. Consider the hull of its support $Y := \mathfrak{Hull}(\{\mathring{x}_1, \ldots, \mathring{x}_n\})$. Again Y is a finite tree, so Y has an endpoint, e.g., $\mathring{x}_1$. Let $z \in \mathcal{X}_0$ be the unique vertex of the edge x_1, such that $\mathring{z} \notin Y$. Clearly, $d_1(\alpha) = \pm\alpha_1 z + \ldots$ has a non-zero coefficient in front of z. This proves that $\alpha = 0$ and d_1 is injective. □

Schneider and Stuhler construct finitely generated projective resolutions as in Conjecture 6.3.7 for connected reductive algebraic groups over non-archimedean local fields by using the action of the group on its Bruhat-Tits building [51], [52]. The Bruhat-Tits building is a contractible simplicial set with very specific properties which allow the conjecture above to hold in full generality. The reason we require a CAT(0)-space is because in the case of groups G with a (B, N)-pair structure and, in particular, complete Kac-Moody groups over finite fields (which we address in the end of Chapter 7), one can construct the so-called *Davis building* of G, which is a CAT(0)-space satisfying the properties requested above.

Chapter 7

Kac-Moody groups

In this chapter we study Kac-Moody groups. More precisely, we are interested in complete Kac-Moody groups defined over a finite field. We begin by recalling what a (B, N)-pair for a group is and how one defines an associated building for such groups. The main references for this are Brown [7], Abramenko and Brown [1]. We then define minimal Kac-Moody groups $G_{\mathfrak{D}}(\mathbb{K})$, where $\mathfrak{D}$ is a root datum associated to a generalised Cartan matrix $\mathcal{A}$ and $\mathbb{K}$ is a field. There are no restrictions on $\mathbb{K}$. For the definition of minimal Kac-Moody groups we follow Carter and Chen [17]. There is also a notion of maximal Kac-Moody groups defined by Kumar, however, we do not work with such groups and so we leave their definition out [39]. We move on to defining our main objects of interest - complete Kac-Moody groups in the case when $\mathbb{K} = \mathbb{F}_q$, the field of $q = p^a$ elements, where p is a prime. We do this in Section 7.2.2. The main references for the complete groups are Caprace-Rémy [15], Rémy-Ronan [45], Capdeboscq-Rumynin [11]. So far the material is well-known. In Section 7.3 we study cocompact lattices in complete rank 2 Kac-Moody groups. The material in this section is original and is taken from a joint paper by Inna Capdeboscq, Dmitriy Rumynin and the author of this thesis [10]. In the final Section 7.4 we study smooth representations of complete Kac-Moody groups and their projective resolutions. We revisit our Theorem 5.3.1 and Theorem 6.2.5 and present their relevant versions for complete Kac-Moody groups over $\mathbb{F}_q$. This material is part of the paper between Dmitriy Rumynin and the author of this thesis [32]. In the last section, the field $\mathbb{F}$

from Chapters 4, 5 and 6 appears again. We explain what restrictions we need to put on its characteristic in order to obtain a Hecke algebra for a complete Kac-Moody group when we need it.

7.1 Groups with (B, N)-pairs and the Bruhat-Tits building

In this section we give a review of groups with a (B, N)-pair structure and their associated buildings. The material is well-known and the main references are [7], [1].

We begin by defining what a group with a (B, N)-pair structure is.

Defintion 7.1.1. [1], [7] We say that a group G admits a (B, N)*-pair structure* (or that (B, N, S) *is a* (B, N)*-pair for* G), if there is a triple (B, N, S), where B and N are subgroups of G, such that $H = B \cap N$ is normal in N, and $N/H = W$ is generated by a set S, where:

1. G is generated by B and N,
2. Every $\mathbf{s} \in S$ has order 2,
3. For $\mathbf{s} \in S$ and $\mathbf{w} \in W$, $\mathbf{s}B\mathbf{w} \subseteq B\mathbf{sw}B \cup B\mathbf{w}B$,
4. $\mathbf{s}B\mathbf{s}^{-1} \nleq B$, for all $\mathbf{s} \in S$.

The group W is called the *Weyl group* associated to the (B, N)-pair. One may also say that the quadruple (G, B, N, S) is a *Tits system.*

Note that (W, S) is a *Coxeter system.* Let $S \coloneqq \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. In particular, this means that the group W admits a presentation:

$$W = \langle \mathbf{s}_1, \ldots, \mathbf{s}_n \mid (\mathbf{s}_i\mathbf{s}_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = 1$ if $i = j$, and $m_{ij} = m_{ji} \in \mathbb{N}_{>0} \cup \infty$ if $i \neq j$. One can construct an $n \times n$ matrix $M = (m_{ij})_{i,j\in\{1,\ldots,n\}}$ called the *Coxeter matrix.*

Let G be a group which admits a (B, N)-pair (B, N, S) with Weyl group W. We have the following standard terminology:

- A subgroup W_J of W is *special* if it is generated by some $J \subset S$.
- A coset $\mathbf{w}W_J$, $\mathbf{w} \in W$, of W_J in W is called a *special coset.*
- If W_J is finite, both W_J and the subset $J \subseteq S$ are called *spherical.*

Note that one can define a partial order by reverse inclusion on the set of special cosets. More precisely, we have $\mathbf{w}_1 W_{J_1} \preceq \mathbf{w}_2 W_{J_2}$, if $J_2 \subseteq J_1$ and $\mathbf{w}_2\mathbf{w}_1^{-1} \subseteq W_{J_1}$. This leads to the following definition:

Defintion 7.1.2. (cf. [1, Definition 3.1]) Let (W, S) be a Coxeter system and $\Sigma(W, S)$ be the poset of special cosets in W ordered by reverse inclusion. Then $\Sigma(W, S)$ is called the *Coxeter complex* associated to (W, S).

Note that one can associate a Coxeter complex as above to every Coxeter system. In particular, every group G which admits a (B, N)-pair structure also has an associated Coxeter complex - the complex associated to its Weyl group W. The main property of interest for us is that $\Sigma(W, S)$ is a simplicial complex of dimension $n - 1$, where $n = |S|$ ([1, Theorem 3.5]). Moreover, clearly by the definition of $\Sigma(W, S)$, the Weyl group W acts on it. The Coxeter complex $\Sigma(W, S)$ is colourable, with every vertex coloured by an element of S, and the action of W on $\Sigma(W, S)$ is type (colour)-preserving [1, Theorem 3.5].

Now let us introduce the notion of a building.

Defintion 7.1.3. ([1, Definition 4.1]) A *building* is a simplicial complex $\mathfrak{B}$ that can be expressed as a union of subcomplexes Σ, called *apartments*, satisfying the following axioms:

1. Each apartment is a Coxeter complex.
2. For any two simplices $\sigma, \tau \in \mathfrak{B}$, there is an apartment Σ containing both of them.

3. If Σ and Σ' are two apartments containing σ and τ, then there is an isomorphism $\varphi : \Sigma \to \Sigma'$ fixing σ and τ pointwise.

Note that condition (3) implies that any two apartments are isomorphic. We can see this by taking σ and τ to be empty [1, 4.1]. In particular, as every apartment is a Coxeter complex, say $\Sigma(W, S)$, where (W, S) is a Coxeter system, and $\dim \Sigma(W, S) = |S| - 1$, we see that all apartments in a building $\mathfrak{B}$ have the same dimension. Therefore, their codimension one faces are also of the same dimension. We call the codimension one faces of an apartment Σ in a building $\mathfrak{B}$ *chambers*. We denote the set of chambers of $\mathfrak{B}$ by $\mathrm{Ch}(\mathfrak{B})$.

Let G be a group which admits a (B, N)-pair (B, N, S) with Weyl group W. We say that:

- A subgroup P_J of G is called *standard parabolic* if it is of the form BW_JB.
- A coset $\mathbf{g}P_J$, $\mathbf{g} \in G$, of P_J in G is called *a standard coset*.
- A subgroup of G is called *parabolic of type J* if it is conjugate to some P_J. It is called *parabolic of finite type* if W_J is spherical.

Similarly to what we did with the special cosets in W, we can order standard cosets in G by reverse inclusion. Call the resulting partially ordered set $\mathcal{BT}$.

Defintion 7.1.4. (cf. [1, 6.2], [7, 3]) Let G be a group that admits a (B, N)-pair structure. The partially ordered set $\mathcal{BT}$ of standard cosets ordered by the opposite of the inclusion order is a building. We refer to this building as *the building associated to the (B, N)-pair* of G.

Before we make further comments, we also recall the following notion:

Defintion 7.1.5. ([1, 6.1.1.]) Let $\mathfrak{B}$ be a building on which a group G acts. The action is called *strongly transitive* if G acts transitively on the set of pairs (Σ, c), where Σ is an apartment of $\mathfrak{B}$ and $c \in \Sigma$ is a chamber of Σ.

Clearly, if G is a group with a (B, N)-pair structure, it acts on its associated building $\mathcal{BT}$. Moreover, one can show that the action is strongly transitive and type-preserving (recall that apartments are colourable, as they are Coxeter complexes).

The condition of strong transitivity allows us to nominate a *fundamental apartment* Σ_0 and a *fundamental chamber* c_0 of $\mathcal{BT}$. What this means is that we fix an arbitrary pair (Σ_0, c_0), where Σ_0 is an apartment in $\mathcal{BT}$ and c_0 is a chamber of Σ_0. By strong transitivity, for any other pair (Σ, c) of an apartment and a chamber in it, there exists a $\mathbf{g} \in G$, such that $(\Sigma, c) = (\mathbf{g}\Sigma_0, \mathbf{g}c_0)$. For the (B,N)-pair (B,N,S), we can choose our fundamental chamber c_0 and fundamental apartment Σ_0, so that we have the following properties:

$$\mathrm{Stab}_G(c_0) = B, \quad \mathrm{Stab}_G(\Sigma_0) = N, \quad \mathrm{Fix}_G(\Sigma_0) = H,$$

where $H = B \cap N$, and Fix_G denotes the pointwise stabiliser (also called the *fixator*, hence the notation). One can observe that stabilisers of vertices are maximal parabolic subgroups of G, i.e., parabolic subgroup of the form $P_i \coloneqq B\langle \mathbf{s}_i \rangle B$, where $\mathbf{s}_i \in S$. More generally, stabilisers of i-simplices in $\mathcal{BT}$ are parabolic subgroups of G of the form BW_JB (or conjugates thereof), where $J \subseteq S$ and $|J| = i$. [7, 3].

Example 7.1.6. (cf. [1], [7]) Let $G :=\mathrm{GL}_2(\mathbb{Q}_p)$, where p is a prime. The building $\mathcal{BT}$ of G has dimension 1. Its vertices are equivalence classes $[\mathcal{L}]$ of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, where $\mathcal{L}$ is a $\mathbb{Z}_p$-lattice and two lattices $\mathcal{L}$ and $\mathcal{L}'$ are equivalent, if there is a $\lambda \in \mathbb{Q}_p^\times$, such that $\mathcal{L}' = \lambda\mathcal{L}$. The edges are families $\{[\mathcal{L}_0], [\mathcal{L}_1]\}$, where $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy the condition

$$\mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq p^{-1}\mathcal{L}_0,$$

where the prime p is the uniformizer in $\mathbb{Q}_p$. One can show that the building is a $(p+1)$-regular tree. The chambers are edges, and the apartments are sequences of edges. Now, observe that G acts transitively on the set of lattices. Let $c_0 \coloneqq \{[\mathbb{Z}_p \oplus \mathbb{Z}_p], [\mathbb{Z}_p \oplus p\mathbb{Z}_p]\}$ be the fundamental chamber of $\mathcal{BT}$. Its stabiliser $\mathrm{Stab}_G(c_0) = B$, where B is the subgroup of G of matrices of the form

$$B = \{\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Q}_p & \mathbb{Z}_p \end{pmatrix}\}.$$

Let N be the set of monomial matrices. The pair (B,N,S) is a (B,N)-pair for G, where $S = \{\mathbf{s}_1, \mathbf{s}_2\}$, where

$$\mathbf{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{s}_2 = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}.$$

The group W is the infinite dihedral group. We have two standard parabolics: $P_1 = B\langle \mathbf{s}_1 \rangle B = BW_{\mathbf{s}_1}B$ and $P_2 = B\langle \mathbf{s}_2 \rangle B = BW_{\mathbf{s}_2}B$. Note that P_1 stabilises the vertex $[\mathbb{Z}_p \oplus \mathbb{Z}_p]$ of c_0 and P_2 stabilises the vertex $[\mathbb{Z}_p \oplus p\mathbb{Z}_p]$. Since parabolics are not conjugate to one another ([7, V. 2B, Theorem 2]), the orbits of these two vertices are disjoint and so they determine the 2 types of vertices in $\mathcal{BT}$. Thus, to summarize the set of vertices in $\mathcal{BT}$ corresponds to cosets of P_1 and P_2 and the set of edges corresponds to cosets of B.

7.2 Minimal and complete Kac-Moody groups

7.2.1 The minimal groups

Minimal Kac-Moody groups are abstract groups defined from a root datum $\mathfrak{D}$ corresponding to a generalised Cartan matrix $\mathcal{A}$, via generators and relations, commonly known as *Tits relations*. Let us define our objects one by one. All the material in this section is well-known and the main references are Tits [53], Carter and Chen [17] and Kumar [39].

Defintion 7.2.1. A *generalised Cartan matrix* $\mathcal{A} = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in \mathbb{Z}$, such that:

- $a_{ii} = 2$ for all $i = 1, .., n$.
- $a_{ij} \leqslant 0$ for all $i, j \in \{1, .., n\}$, $i \neq j$.
- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Defintion 7.2.2. A *root datum of type* $\mathcal{A}$ is a quintuple $\mathfrak{D} = (\mathcal{A}, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$, where

- $\mathcal{A}$ is an $n \times n$ generalised Cartan matrix,

- $\mathcal{Y}$ is a finitely generated free abelian group of rank k,
- $\mathcal{X} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{Y}, \mathbb{Z})$ is its dual group,
- $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{X}$ is a set whose elements are called simple roots,
- $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subseteq \mathcal{Y}$ is a set whose elements are called simple coroots,

satisfying a single axiom $\alpha_i(\alpha_j^\vee) = a_{ji}$ for all $i, j \in \{1, .., n\}$.

We clearly have $n \leqslant k$. Since the elements $\alpha_i^\vee$, $i \in \{1, \ldots, n\}$, do not have to form a basis of $\mathcal{Y}$. We need to overcome this issue in order to obtain the Weyl group W of the generalised Cartan matrix $\mathcal{A}$. Let

$$\widetilde{\Pi} = \{\tilde{\alpha_1}, \ldots, \tilde{\alpha_n}\}$$

be a set. Define $\mathcal{Z}$ to be the free abelian group of rank n generated by $\widetilde{\Pi}$. Let $\mathbf{s}_i$ be the automorphism of $\mathcal{Z}$ given by

$$\mathbf{s}_i : \mathcal{Z} \to \mathcal{Z}, \quad \tilde{\alpha_j} \mapsto \alpha_j - a_{ij}\tilde{\alpha}_i.$$

Let $S = \{\mathbf{s}_1, \ldots \mathbf{s}_n\}$. The group W generated by S is called the *Weyl group of* $\mathcal{A}$. The set $\Phi = W(\widetilde{\Pi})$ is called the *set of real roots*. Every $\alpha \in \Phi$ can be written as $\alpha = \sum_i \lambda_i \tilde{\alpha}_i$, with $\lambda_i \in \mathbb{Z}$ for all i. It is a standard fact that either $\lambda_i \geqslant 0$ for all i, or $\lambda_i \leqslant 0$ for all i. Thus, we have $\Phi = \Phi_+ \cup \Phi_-$, where $\Phi_+ = \{\alpha \in \Phi \mid \lambda_i \geqslant 0\}$ called the *set of positive roots*, and $\Phi_- = \{\alpha \in \Phi \mid \lambda_i \leqslant 0\}$, called the *set of negative roots*.

Let $\mathbb{K}$ be a field. For every $\alpha \in \Phi$ and every $t \in \mathbb{K}$, introduce the symbol $\mathbf{x}_\alpha(t)$.

Defintion 7.2.3. (cf. [17]) To every real root $\alpha \in \Phi$, associate an additive group U_α defined as follows:

- U_α is generated by the symbols $\mathbf{x}_\alpha(t)$, $t \in \mathbb{K}$ defined above;
- $\mathbf{x}_\alpha(t)\mathbf{x}_\alpha(s) = \mathbf{x}_\alpha(t + s)$, for all $t, s \in \mathbb{K}$.

The groups U_α are called *root groups*.

Note that $U_\alpha \cong (\mathbb{K}, +)$. Next we define the group

$$H \coloneqq \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{K}^\times.$$

The Weyl group W acts on $\mathcal{Z}$ by definition. It also acts on $\mathcal{Y}$:

$$\mathbf{s}_i(\mathbf{y}) = \mathbf{y} - \mathbf{y}(\alpha_i)\alpha_i^\vee$$

and hence on H:

$$\mathbf{s}_i(\mathbf{h}) = \mathbf{s}_i(\mathbf{y} \otimes t) = \mathbf{s}_i(\mathbf{y}) \otimes t, \text{ for all } \mathbf{y} \in \mathcal{Y}, t \in \mathbb{K}.$$

For every $t \in \mathbb{K}$ define the following elements:

$$\mathbf{x}_i(t) = \mathbf{x}_{\alpha_i}(t), \quad \mathbf{x}_{-i}(t) = \mathbf{x}_{-\alpha_i}(t)$$

$$\tilde{\mathbf{s}_i}(t) = \mathbf{x}_i(t)\mathbf{x}_{-i}(t^{-1})\mathbf{x}_i(t), \quad \tilde{\mathbf{s}_i} = \tilde{\mathbf{s}_i}(1)$$

$$\mathbf{h}_i(t) = \tilde{\mathbf{s}_i}(t)\tilde{\mathbf{s}_i}^{-1}.$$

Defintion 7.2.4. The *minimal Kac-Moody group* $G \coloneqq G_{\mathfrak{D}}(\mathbb{K})$ associated to a root datum $\mathfrak{D} = (\mathcal{A}, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$ of type $\mathcal{A}$ and a field $\mathbb{K}$, is the quotient of the free product of the group $H = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{K}^\times$ and the root groups U_α, for $\alpha \in \Phi$, by the relations:

1. $\mathbf{h}_i(t) = \alpha_i^\vee \otimes t$,
2. $(\mathbf{y} \otimes t)\mathbf{x}_i(s)(\mathbf{y} \otimes t)^{-1} = \mathbf{x}_i(t^{\alpha_i(\mathbf{y})}s)$,
3. $\tilde{\mathbf{s}_i}(\mathbf{y} \otimes t)\tilde{\mathbf{s}_i}^{-1} = \mathbf{s}_i(y) \otimes t$,
4. $\tilde{\mathbf{s}_i}\mathbf{x}_\alpha(t)\tilde{\mathbf{s}_i}^{-1} = \mathbf{x}_{\mathbf{s}_i}(\epsilon_{i,\alpha}t)$, for a uniquely determined $\epsilon_{i,\alpha} \in \{-1, 1\}$,
5. For every $\alpha, \beta \in \Phi$, such that there exist $\mathbf{w}, \mathbf{w}' \in \Phi$ with $\mathbf{w}(\alpha), \mathbf{w}(\beta) \in \Phi_+$ and

$\mathbf{w}'(\alpha), \mathbf{w}'(\beta) \in \Phi_-$

$$[\mathbf{x}_\alpha(t), \mathbf{x}_\beta(s)] = \prod_{i,j \in \mathbb{Z}_{>0}, i\alpha + j\beta \in \Phi} \mathbf{x}_{i\alpha+j\beta}(C_{ij\alpha\beta} t^i s^j), \text{ for } t, s \in \mathbb{K},$$

for integers $C_{ij\alpha\beta}$ uniquely determined by i, j, α, β and $\mathfrak{D}$.

A minimal Kac-Moody group G is called *simply connected* if $k = n$ and $\Pi^\vee$ is a basis for $\mathcal{Y}$. In this case $H \cong (\mathbb{K}^\times)^n$. The subgroup H is called *the torus* of G.

Let $G = G_{\mathfrak{D}}(\mathbb{K})$ be a minimal Kac-Moody group associated to some root datum $\mathfrak{D}$ of type $\mathcal{A}$ and a field $\mathbb{K}$. Let U be the subgroup of G defined by

$$U \coloneqq U_+ = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle.$$

Consider the following subgroups of G

$$B = U \rtimes H, \qquad N = \langle \tilde{\mathbf{s}_1}, ..., \tilde{\mathbf{s}_n}, H \rangle.$$

Note that there is a group homomorphism $\pi : N \to W$, given by $\tilde{\mathbf{s}_i} \mapsto \mathbf{s}_i$. By the defining relations of a Kac-Moody group, we know that N acts on H by conjugation. Thus, $H \trianglerighteq N$, and π gives rise to a homomorphism $\bar{\pi} : N/H \to W$, given by $\tilde{\mathbf{s}}_i H \mapsto \mathbf{s}_i$. This is an isomophism and so $W \cong N/H$ [39]. Note that also $B \cap N = H$. Thus, the triple (B, N, S) is a (B, N)-pair for G.

7.2.2 Complete groups

The material in this section is well-known. The main references are [16], [15], [11], [42], [45].

Having defined the minimal Kac-Moody group $G_{\mathfrak{D}}(\mathbb{K})$ over a field $\mathbb{K}$, we would like to look at completions of it. More precisely, we will endow G with a topology, and look at topological completions of this. If we are not referring to a specific completion, we denote the complete group by $\widetilde{G}$. We are interested in the cases when $\widetilde{G}$ is a locally compact totally disconnected group. This can happen in several completions, but they all require the underlying field $\mathbb{K}$ to be finite. So from now on, let $\mathbb{K} = \mathbb{F}_q$, where $q = p^a$, for some prime p. The completions which yield a locally compact group are:

- the *Carbone-Garland completion* with respect to the weight topology,
- the *Rémy-Ronan completion* obtained by embedding G into $\mathrm{Aut}(\mathcal{BT})$, where $\mathcal{BT}$ is the building of G (as defined in Section 7.1 for a group with a (B, N)-pair structure),
- the *Caprace-Rémy completion* with respect to the *building topology*,
- *the local pro-p completion* defined by Capdeboscq-Rumynin.

We will not be interested in the Carbone-Garland completion, as the weight topology requires us passing to the Kac-Moody algebra, so this does not suit our abstract group-theoretic setting. We will mainly concentrate on the Caprace-Rémy and the local pro-p completions, so we will define these now. As the Rémy-Ronan completion also fits our setting we give its definition too.

Let $G = G_{\mathfrak{D}}(\mathbb{F}_q)$ be a minimal Kac-Moody group associated to a root datum $\mathfrak{D}$ and the field $\mathbb{F}_q$. In the end of the previous section we explained that minimal Kac-Moody groups admit a (B, N)-pair structure. By Section 7.1 we know that every group with a (B, N)-pair has an associated building on which it acts. Thus, denote by $\mathcal{BT}$ the building associated to $G = G_{\mathfrak{D}}(\mathbb{F}_q)$. Let K the kernel of the G-action on $\mathcal{BT}$. Then $K = Z(G)$, where $Z(G)$ denotes the centre of G. When G is defined over $\mathbb{F}_q$, the centre $Z(G)$ is a finite group [45]. Let $\mathrm{Aut}(\mathcal{BT})$ be the automorphism group of $\mathcal{BT}$. We have an embedding:

$$\varphi : G/K \hookrightarrow \mathrm{Aut}(\mathcal{BT}).$$

As a topological group $\mathrm{Aut}(\mathcal{BT})$ is locally compact [45], [2]. Thus, we can endow G/K with a topology induced by the map φ.

Defintion 7.2.5. [45] The *Rémy-Ronan completion of* G, is the topological group $G^{rr} = \varphi\overline{(G/K)}$, i.e., it is the closure of $\mathrm{im}(\varphi)$ in $\mathrm{Aut}(\mathcal{BT})$.

The group G^{rr} is locally compact totally disconnected [45]. Now we move on to the Caprace-Rémy completion. Recall that $Ch(\mathcal{BT})$ denotes the set of chambers of $\mathcal{BT}$. Let c_0 be the fundamental chamber of $\mathcal{BT}$. From Section 7.1 we know that $\mathrm{Stab}_G(c_0) = B$. For each $n \in \mathbb{N}$ define

$$U_{+,n} := \{\mathbf{g} \in U \mid \mathbf{g} \cdot c' = c' \text{ for every } c' \in Ch(\mathcal{BT}), \text{ such that } d(c_0, c') \leqslant n\},$$

where d denotes the distance function on $\mathcal{BT}$ [7],[15]. Define the following left-invariant metric $d_+ : G \times G \to \mathbb{R}_+$ on G:

$$d_+(\mathbf{g}, \mathbf{h}) = \begin{cases} 2, \text{ if } \mathbf{g}^{-1}\mathbf{h} \notin U \ , \\ 2^{-n}, \text{ if } \mathbf{g}^{-1}\mathbf{h} \in U \ \text{ and } \ n = \max\{k \in \mathbb{N} \mid \mathbf{g}^{-1}\mathbf{h} \in U_{+,k}\} \end{cases}$$

for all $\mathbf{g}, \mathbf{h} \in G$ [15].

Defintion 7.2.6. [15] The topology obtained from this metric is called the *building topology* on G. The completion of the topological space G with respect to the building topology is a group $\overline{G}$ and is called the *Caprace-Rémy completion* of G.

Caprace and Rémy show that the topological group $\overline{G}$ is locally compact totally disconnected with (B, N)-pair $(\overline{B}, N, S)$ [15, Proposition 1].

Finally we introduce the locally pro-p-complete group $\widehat{G}$. Let

$$\mathcal{F} := \{A \leqslant U \mid |U : A| = p^k, \text{ for some } k \in \mathbb{N}\}$$

be a set of subgroups of B. It forms a fundamental system of neighbourhoods of 1 in B. The completion $\widehat{B}$ of B in this topology is a group and this topology also defines a topology on G [11, Theorem 1.2]. The resulting group is denoted $\widehat{G}$.

Defintion 7.2.7. The *local pro-p completion of* G is the topological group $\widehat{G}$ described above.

The group $\widehat{G}$ is locally compact and totally disconnected, with (B, N)-pair $(\widehat{B}, N, S)$. Moreover, $\widehat{B}$ is open in $\widehat{G}$ and the completion $\widehat{U}$ of U is a pro-p group

[11].

The completions described above are related as follows:

Proposition 7.2.8. *[11], [15], [49] There is a sequence of open continuous surjective homomorphisms of topological groups:*

$$\widehat{G} \twoheadrightarrow \overline{G} \twoheadrightarrow G^{rr}.$$

7.3 Cocompact lattices in rank 2 Kac-Moody groups

To the best of my knowledge, all the results in this section are original and are taken from a joint paper between Inna Capdeboscq, Dmitriy Rumynin, and the author of this thesis [10]. In this section we study cocompact lattices in complete rank 2 Kac-Moody groups defined over $\mathbb{F}_q$. More precisely, we study cocompact lattices in the locally pro-p complete Kac-Moody groups $\widehat{G}$ by relating them to the already classified by Capdeboscq-Thomas edge transitive cocompact lattices in the Caprace-Rémy complete Kac-Moody groups $\overline{G}$ [12]. The reason why we study cocompact lattices in rank 2 only is that in rank n, with $n > 2$, there are no cocompact lattices, except in a certain special case already described by Caprace-Monod [14]. Other known lattices in rank 2 groups include lattices discovered by Capdeboscq-Thomas [13], Carbone-Garland [16], Rémy-Ronan [45], Gramlich-Horn-Mühlherr [28].

7.3.1 Cocompact lattices in $\widehat{G}$

We concentrate on rank 2 Kac-Moody groups over a finite field $\mathbb{F}_q$, where $q = p^a$, for some prime p. Let G denote the minimal such Kac-Moody group. Keeping the notation as in Section 7.2 $G = G_{\mathfrak{D}}(\mathbb{F}_q)$, where $\mathfrak{D}$ is a root datum of type $\mathcal{A}$ and $\mathcal{A}$ is a generalised Cartan matrix. Since G is of rank 2, $\mathcal{A}$ is a 2×2 matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix},$$

with $a_{12}, a_{21} \in \mathbb{Z}_{\leqslant 0}$. We make the additional assumption that $\max(a_{21}, a_{12}) \leqslant -2$. As before we have groups $\mathcal{X}$, $\mathcal{Y}$ and sets Π and $\Pi^\vee$, where $\mathcal{Y}$ is a finitely generated free abelian group, $\mathcal{X} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{Y}, \mathbb{Z})$ is its dual group, $\Pi = \{\alpha_1, \alpha_2\} \subset \mathcal{X}$ is the set of simple roots and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee\} \subset \mathcal{Y}$ the set of simple coroots. Additionally, $\alpha_i(\alpha_j^\vee) = a_{ji}$ for all $i, j \in \{1, 2\}$. The Weyl group of G in this case is the infinite dihedral group

$$W = \langle \mathbf{s}_1, \mathbf{s}_2 \mid \mathbf{s}_1^2 = \mathbf{s}_2^2 \rangle.$$

Consequently we have the set of real roots $\Phi = W(\Pi) = \{\mathbf{w}\alpha_1, \mathbf{w}\alpha_2 \mid \mathbf{w} \in W\}$. As before $\Phi = \Phi_+ \cup \Phi_-$, where Φ_+ is the set of positive real roots and Φ_- is the set of negative real roots. In the current setting, the set Φ_+ can be decomposed further [12]:

$$\Phi_+ = \Phi_+^1 \cup \Phi_+^2,$$

where the union is disjoint and

$$\Phi_+^1 := \{\alpha_1, \mathbf{s}_1\alpha_2, \mathbf{s}_1\mathbf{s}_2\alpha_1, \mathbf{s}_1\mathbf{s}_2\mathbf{s}_1\alpha_2, ..., [\mathbf{s}_1\mathbf{s}_2]_n\alpha_1, [\mathbf{s}_1\mathbf{s}_2]_n\mathbf{s}_1\alpha_2, ...\},$$

$$\Phi_+^2 := \{\alpha_2, \mathbf{s}_2\alpha_1, \mathbf{s}_2\mathbf{s}_1\alpha_2, \mathbf{s}_2\mathbf{s}_1\mathbf{s}_2\alpha_1, ..., [\mathbf{s}_2\mathbf{s}_1]_n\alpha_2, [\mathbf{s}_2\mathbf{s}_1]_n\mathbf{s}_2\alpha_1, ...\},$$

where $[\mathbf{s}_i\mathbf{s}_j]_m := \mathbf{s}_i\mathbf{s}_j\mathbf{s}_i \ldots$ (there are m symbols on the right hand side). We also have the sets $-\Phi_+^1 := \{-\alpha \mid \alpha \in \Phi_+^1\}$ and $-\Phi_+^2 := \{-\alpha \mid \alpha \in \Phi_+^2\}$.

The root subgroups are $U_\alpha \cong (\mathbb{F}_q, +)$. As in Section 7.2.1 we have the subgroups

$$B = U \rtimes H, \quad N = \langle \tilde{\mathbf{s}_1}, \tilde{\mathbf{s}_2}, H \rangle$$

with the triple (B, N, S), where S is the generating set of the Weyl group W of G, forming a (B, N)-pair for G. We also have standard maximal parabolic subgroups of G:

$$P_i := B\langle \tilde{\mathbf{s}_i} \rangle B, \quad \text{for } i = 1, 2,$$

where $\tilde{\mathbf{s}_i} = \bar{\pi}^{-1}(\mathbf{s}_i)$. Similar to Example 7.1.6, in the current setting the building $\mathcal{BT}$ of G is a $(q+1)$-regular tree with vertices corresponding to the G-conjugates of the parabolics P_i, $i = 1, 2$, and edges corresponding to conjugates of B [16]. Denote the set of vertices by $\mathcal{BT}_0$ and the set of edges by $\mathcal{BT}_1$. Again similar to Example 7.1.6, there are two types of vertices in $\mathcal{BT}_0$ - one corresponding to cosets of P_1, and the second one corresponding to cosets of P_2 [16], [12].

In Section 7.2 we explained there are different topologies one can put on a Kac-Moody group to make it into a topological group. We would be interested in the Caprace-Rémy completion $\overline{G}$ of G, and in the local pro-p completion $\widehat{G}$. Both groups $\overline{G}$ and $\widehat{G}$ are locally compact totally disconnected. If we want to discuss the two groups in parallel, we will use the notation $\widetilde{G}$. The sets Φ^1_+ and Φ^2_+ defined above give rise to the following elementary abelian p-subgroups of G:

$$U_i := \langle U_\alpha \mid \alpha \in \Phi^i_+ \rangle \text{ and } -U_i := \langle U_\alpha \mid -\alpha \in \Phi^i_+ \rangle, \text{ for } i = 1, 2.$$

The building $\mathcal{BT}$ of G has a unique end ϵ_i determined by the simple root α_i, $i \in \{1, 2\}$ [12]. The groups U_1 and $-U_2$ fix ϵ_1 and U_2 and $-U_1$ fix ϵ_2 [12]. We also have the properties $[U_1, -U_2] = 1$ and $U_1 \cap -U_2 = 1$. The same holds true for U_2 and $-U_1$ [12]. Consider the subgroups of $\widetilde{G}$

$$\mathcal{U}_1 := \mathrm{cl}(U_1 \times -U_2) \text{ and } \mathcal{U}_2 := \mathrm{cl}(-U_1 \times U_2),$$

where cl( ) denotes the closure in the relevant topology. Note that $\mathcal{U}_2 = \mathbf{s}_1 \mathcal{U}_1 \mathbf{s}_1^{-1}$. As from now on we will only be interested in conjugates of $\mathcal{U}_1$, without loss of generality we fix $\mathcal{U} := \mathcal{U}_1$.

Recall the following definition:

Defintion 7.3.1. ([12], [2]) Let H be a locally compact group with a left Haar measure μ (note that μ is not normalised). A *lattice* Γ in H is a discrete subgroup $\Gamma \leqslant H$, such that $\Gamma \backslash H$ has a left H-invariant Haar measure. Furthermore, a lattice $\Gamma \leqslant H$ is called *cocompact* if $\Gamma \backslash H$ is compact.

To inverstigate cocompact lattices in a complete Kac-Moody group $\widetilde{G}$ we want the elemets of order p in $\widetilde{G}$ to satisfy certain properties. We propose the following definition:

Defintion 7.3.2. We say that a complete Kac-Moody group $\widetilde{G}$ is *p-well-behaved*, if the following conditions hold:

(P1) Cocompact lattices in $\widetilde{G}$ do not contain p-elements.

(P2) Any element of order p in $\widetilde{G}$ is contained in a conjugate of the subgroup $\mathcal{U}$.

The first main aim of this section is to prove that the local pro-p completion $\widehat{G}$ of G is p-well-behaved. It is an open conjecture that $\overline{G}$ is p-well-behaved [12].

Lemma 7.3.3. *Let $U = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle$, and U_1 and U_2 be as above. Then U is a free product of U_1 and U_2.*

Proof. By [54, Proposition 4], U is an amalgamated product of U_1 and U_2 along their intersection $U_0 = U_1 \cap U_2$. However, as remarked above $U_0 = 1$. The result follows. □

As the name suggests, the local pro-p completion is a local version of the pro-p completion of a group. However, for the subgroup $U \leqslant G$ it turns out that its local pro-p completion $\widehat{U}$ is in fact its full pro-p completion [11]. Ribes and Zalesskii define the following product for pro-p groups:

Defintion 7.3.4. [47, 9.1] Let $\{G_i\}_{i \in \mathcal{I}}$ be a collection of pro-p groups. Then their *free pro-p product* is a pro-p group G and continuous homomorphisms $\varphi_i : G_i \to G$, such that for any pro-p group K and any continuous homomorphisms $\psi_i : G_i \to K$, there exists a unique continuous homomorphism $\psi : G \to K$ making the diagrams commute:

$$\begin{tikzcd} G \arrow[dr, dashed, "\psi"] & \\ G_i \arrow[u, "\varphi_i"] \arrow[r, "\psi_i"'] & K \end{tikzcd}$$

We denote the free pro-p product by $\amalg$.

Theorem 7.3.5. *Any element of order p in $\widehat{G}$ is contained in a conjugate of the subgroup $\mathcal{U}$ of $\widehat{G}$. In particular, $\widehat{G}$ satisfies (P2).*

Proof. Let $\mathbf{g} \in \widehat{G}$ be an element of order p. Then $\mathbf{g}$ lies in a conjugate of the Sylow pro-p subgroup $\widehat{U}$ of $\widehat{G}$ [11]. Thus, without loss of generality we may assume that $\mathbf{g} \in \widehat{U}$. By Lemma 7.3.3, $U = U_1 * U_2$ and, thus, $\widehat{U} = \widehat{U_1 * U_2}$. As taking a pro-$p$ completion commutes with taking a free pro-p product we have [47, 9.1.1]:

$$\widehat{U} = \widehat{U_1 * U_2} \cong \widehat{U_1} \amalg \widehat{U_2}.$$

Note that the completions $\widehat{U_i}$ are profinite groups. Also as $U_i \leqslant U$, then $\mathrm{cl}(U_i) \leqslant \widehat{U}$ and thus $\mathrm{cl}(U_i)$ are closed subgroups of a profinite group, hence profinite. Thus $\mathrm{cl}(U_i) \cong \widehat{U_i}$, for $i = 1, 2$. Hertfort and Ribes [29] show that if a group decomposes as a free pro-p product of two groups, then all the torsion is contained in a conjugate of one of the factors. Hence, $\mathbf{g}$ is contained in a conjugate of one of $\widehat{U_i}$, $i = 1, 2$. Since $\mathcal{U} = \mathrm{cl}(U_1 \times -U_2)$, the proof is now complete. □

Our next step is to explain why in $\widehat{G}$ Property (P2) implies Property (P1). We begin our investigation with a lemma.

Lemma 7.3.6. *Let G be a minimal Kac-Moody group of rank 2 over $\mathbb{F}_q$, $\widehat{G}$ its local pro-p completion and $\overline{G}$ its Caprace-Rémy completion. Let*

$$C := C(\widehat{G}, \widehat{U}) = \bigcap_{g \in \widehat{G}} g\widehat{U}g^{-1} .$$

Then

$$\widehat{G}/C \cong \overline{G}$$

as topological groups.

Proof. Recall that for two groups to be isomorphic as topological groups, we need to show existence of a continuous abstract group isomorphism with a continuous inverse. Let $\pi : \widehat{G} \twoheadrightarrow \overline{G}$ be the natural map. This is an open continuous homomorphism with kernel $\ker(\pi) = C$ [11]. Consider the group $\widehat{G}/C$ as a topological group with respect to the quotient topology coming from $\widehat{G}$. We have a commutative diagram

$$
\begin{tikzcd}
\widehat{G} \arrow[r, "\pi"] \arrow[dr, "\theta"'] & \overline{G} \\
& \widehat{G}/C \arrow[u, "\bar{\pi}"]
\end{tikzcd}
,
$$

where θ is the quotient map and $\bar{\pi}$ is the natural map induced by π and factoring through θ. Note that since $\ker(\pi) = C$, $\bar{\pi}$ is in fact an isomorphism of abstract groups. Moreover, the map θ is a continuous, open surjection [30, 5.16, 5.17]. Since θ is surjective, it has a left inverse, i.e., a map $h : \widehat{G}/C \to \widehat{G}$, such that $\theta \circ h = \mathrm{id}_{\widehat{G}/C}$. Take $N \subseteq \overline{G}$ to be an open subset, then

$$\pi^{-1}(N) = \theta^{-1}\big(\bar{\pi}^{-1}(N)\big) = h\big(\bar{\pi}^{-1}(N)\big).$$

Thus,

$$\theta\big(\pi^{-1}(N)\big) = \theta \circ h\big(\bar{\pi}^{-1}(N)\big) = \bar{\pi}^{-1}(N).$$

Since π is continuous and θ is open, $\bar{\pi}^{-1}(N)$ is open, and so $\bar{\pi}$ is continuous. Furthermore, $\bar{\pi}$ also has a continuous inverse. Let $\psi := \bar{\pi}^{-1}$. Take an open $L \subseteq \widehat{G}/C$. We have

$$\psi^{-1}(L) = \bar{\pi}(L) = \bar{\pi}\big(\theta \circ h(L)\big) = \pi \circ h(L) = \pi \circ \theta^{-1}(L).$$

By continuity of θ and openness of π, $\psi^{-1}(L)$ is open, finishing the proof. $\square$

To establish Property (P1) for $\widehat{G}$ we need to work a little harder. Recall that a topological space is *first countable* if every point has a countable basis of neighbourhoods. A topological space X is *metrizable* if there exists a metric d on X, such that (X, d) is a metric space and the topology on X is induced by the metric. We make the following claim for our complete Kac-Moody group $\widehat{G}$:

Proposition 7.3.7. *The group $\widehat{G}$ is metrizable.*

Proof. A topological group is metrizable if and only if it is first countable [30, 8.3]. Since left and right translations in a topological group are homeomorphisms [30, 4.2] and $\widehat{U}$ is open in $\widehat{G}$, to show first countability of $\widehat{G}$, it is enough to show it for $\widehat{U}$. Recall that $\widehat{U}$ is the full pro-p completion of U. Thus, it is a profinite group and a profinite group is first countable if it admits a countably infinite generating set which converges to 1 [47, Remark 2.6.7]. By Lemma 7.3.3,

$$\widehat{U} = \widehat{U_1 * U_2}\ .$$

Since U_i is an elementary abelian p-group, it is a countably dimensional vector space over the field $\mathbb{F}_q$. Its $\mathbb{F}_p$-basis $e_k^{(i)}$ forms a countable generating sequence, converging to 1 in the pro-p-topology. It follows that $e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)} \ldots$ is a countable generating sequence of $\widehat{U}$, converging to 1, finishing the proof. □

The main point for us is that in a metrizable topological space X compactness is equivalent to *sequential compactness*, in particular, every sequence in X has a convergent subsequence, whose limit lies in X.

Proposition 7.3.8. *(cf. [12, Cor. 4.3.])*

Let $\mathbf{u}$ *be an element of* $\mathcal{U}$*. Then there exists* $\mathbf{g} \in \widehat{G}$*, such that the sequence*

$$x_n := \mathbf{g}^n \mathbf{u} \mathbf{g}^{-n}\ ,\ \ n \in \mathbb{N},$$

has a limit point in C*, where* $C = \ker(\pi)$ *with* $\pi : \widehat{G} \to \overline{G}$ *as in Lemma 7.3.6.*

Proof. Capdeboscq and Thomas prove ([12, Cor. 4.2]) that if $\mathbf{u}$ is any non-trivial element of any $\overline{G}$-conjugate of $\mathcal{U}$, there exists an $\mathbf{h} \in \overline{G}$, such that

$$\lim_{n\to\infty} \mathbf{h}^n \mathbf{u} \mathbf{h}^{-n} = 1_{\overline{G}}.$$

Let $\pi : \widehat{G} \to \overline{G}$ be the homomorphism from Lemma 7.3.6. Using [12, Cor. 4.2] and the surjectivity of π, we know there exists a $\mathbf{g} \in \widehat{G}$, such that

$$\lim_{n\to\infty} \pi(x_n) = \lim_{n\to\infty} \pi(\mathbf{g})^n \pi(\mathbf{u}) \pi(\mathbf{g})^{-n} = 1_{\overline{G}}.$$

Take a compact open subgroup $K \leqslant \widehat{G}$. The map π is open, so $\pi(K)$ is an open neighbourhood of $1_{\overline{G}}$. Thus, there exists $N \in \mathbb{N}$, such that $\pi(x_n) \in \pi(K)$, for all $n > N$. As $\overline{G} \cong \widehat{G}/C$, $x_n \in KC$, for all $n > N$. Since $C \leqslant \widehat{U} \leqslant \widehat{G}$, and $\widehat{U}$ is a compact (since it is profinite), open and closed (since every open subgroup is also closed) subgroup of G, C is a closed compact pro-p subgroup of $\widehat{G}$. Thus, as a product of compact sets KC is compact. By sequential compactness (x_n) contains a convergent subsequence (y_n) in KC. Its limit $\mathbf{z} = \lim_{n\to\infty} y_n$ must belong to C because $\pi(\mathbf{z}) = \lim_{n\to\infty} \pi(y_n) = \lim_{n\to\infty} \pi(x_n) = 1_{\overline{G}}$ and $C = \ker(\pi)$. □

We are nearly ready to prove property (P1) for $\widehat{G}$. The final ingredient we need is the following result for cocompact lattices:

Lemma 7.3.9. *(cf. [26, p. 10]) Let $\Gamma \leqslant \widehat{G}$ be a cocompact lattice. For each $\mathbf{u} \in \Gamma$ its conjugacy class*

$$u^{\widehat{G}} \;=\; \{\mathbf{g}^{-1}\mathbf{u}\mathbf{g} \mid \mathbf{g} \in \widehat{G}\}$$

is a closed subset of $\widehat{G}$.

Proof. Let us show that Γ admits a compact fundamental domain $\widetilde{K}$ in $\widehat{G}$, i.e., a compact subset $\widetilde{K}$ such that $\widehat{G} = \Gamma\widetilde{K}$. Take a compact open subgroup $K \leqslant \widehat{G}$. Consider the quotient map $\theta : \widehat{G} \twoheadrightarrow \Gamma\backslash\widehat{G}$. For each $\mathbf{x} \in \widehat{G}$ the set $\Gamma\mathbf{x}K$ is open and Γ-equivariant. By the definition of the quotient topology, $\theta(\mathbf{x})K = \theta(\Gamma\mathbf{x}K)$ is open in $\Gamma\backslash\widehat{G}$. Thus, $\{\theta(\mathbf{x})K \mid \mathbf{x} \in \widehat{G}\}$ is an open cover of $\Gamma\backslash\widehat{G}$. But Γ is cocompact, so we can choose a finite subcover $\{\theta(\mathbf{x}_i)K \mid i = 1, \ldots, n\}$. It follows that $\widetilde{K} := \bigcup_{i=1}^{n} \mathbf{x}_i K$ is a compact fundamental domain.

The rest of the argument follows Gelfand, Graev and Piatetsky-Shapiro [26, p. 10]. Take $\mathbf{x} \in \mathrm{cl}(u^{\widehat{G}})$. Since $\widehat{G}$ is first countable, there exists a sequence $(\mathbf{g}_i^{-1}\mathbf{u}\mathbf{g}_i)$ with $\mathbf{g}_i \in \widehat{G}$, $i \in \mathbb{N}$, convergent to $\mathbf{x}$. Since $\widehat{G} = \widetilde{K}\Gamma$, we can write each $\mathbf{g}_i$ as $\mathbf{u}_i\mathbf{k}_i$ for some $\mathbf{u}_i \in \Gamma$, $\mathbf{k}_i \in \widetilde{K}$. Since $\widetilde{K}$ is compact and first countable, we can choose a convergent subsequence of $(\mathbf{k}_i)$. Thus, without loss of generality $(\mathbf{k}_i)$ converges to some $\mathbf{k} \in \widetilde{K}$. Observe that

$$\mathbf{u}_i^{-1}\mathbf{u}\mathbf{u}_i = \mathbf{k}_i\mathbf{g}_i^{-1}\mathbf{u}\mathbf{g}_i\mathbf{k}_i^{-1} = \mathbf{k}_i(\mathbf{g}_i^{-1}\mathbf{u}\mathbf{g}_i)\mathbf{k}_i^{-1} \longrightarrow \mathbf{k}(\lim \mathbf{g}_i^{-1}\mathbf{u}\mathbf{g}_i)\mathbf{k}^{-1} = \mathbf{k}\mathbf{x}\mathbf{k}^{-1}\ .$$

Since Γ is discrete, $\mathbf{u}_n^{-1}\mathbf{u}\mathbf{u}_n = \mathbf{k}\mathbf{x}\mathbf{k}^{-1}$ for all sufficiently large n, so that $\mathbf{x} \in u^{\widehat{G}}$. □

Lemma 7.3.10. *With notation as in Lemma 7.3.9, the set $\pi(u)^{\overline{G}}$ is closed in $\overline{G}$.*

Proof. Observe that, as C is compact, the quotient map $\theta : \widehat{G} \to \widehat{G}/C$ is closed [30, 5.18]. It follows from Lemma 7.3.6 that the natural map $\pi : \widehat{G} \to \overline{G}$ is closed. Thus, the set $\pi(u^{\widehat{G}}) = \pi(u)^{\overline{G}}$ is closed for every $\mathbf{u}$ from any cocompact lattice Γ. $\square$

Theorem 7.3.11. *A cocompact lattice in $\widehat{G}$ does not contain elements of order p.*

Proof. Let $\Gamma \leqslant \widehat{G}$ be a cocompact lattice. Consider $\mathbf{u} \in \Gamma$, such that $\mathbf{u}$ has order p. By the proof of Theorem 7.3.5, $\mathbf{u}$ is contained in a conjugate of $\widehat{U_1}$ or $\widehat{U_2}$. Without loss of generality, $\mathbf{u} \in \widehat{U_1}$. By Proposition 7.3.8 there exists a $\mathbf{g} \in \widehat{G}$, such that the sequence

$$x_n := \mathbf{g}^n \mathbf{u} \mathbf{g}^{-n}, \quad n \in \mathbb{N},$$

has a limit point $\mathbf{x} \in C$. By construction, the sequence (x_n) lies in the closed set $u^{\widehat{G}}$. Thus $\mathbf{x} \in u^{\widehat{G}} \cap C$. Since C acts trivially on the Bruhat-Tits building of G, so does $\mathbf{x}$. This is a contradiction: elements of $\widehat{U_1}$ do not act trivially on the Bruhat-Tits building of G - they only fix the end ϵ_1. $\square$

Theorem 7.3.11 and Theorem 7.3.5 tell us that:

Corollary 7.3.12. *$\widehat{G}$ is p-well-behaved.*

There is a surprising connection between lattices in $\widehat{G}$ and $\overline{G}$. In general, if we are given two locally compact groups and a continuous homomorphism $\varphi : H \to H'$, there is no reason why for a lattice $\Gamma \leqslant H$, its image $\varphi(\Gamma)$ should be a lattice of H'. The same statement is true if we take a lattice $\Gamma' \leqslant H'$ and look at its preimage under φ. However, for $\widehat{G}$ and $\overline{G}$ we can push and pull lattices along the map π defined in Lemma 7.3.6. Let us begin with our *pushing* procedure.

Proposition 7.3.13. *Let $\theta : H \to H'$ be a continuous homomorphism of locally compact groups, such that $K := \ker(\theta)$ is compact and the image is cocompact and closed. If Γ is a cocompact lattice in H, then $\theta(\Gamma)$ is a cocompact lattice in H'.*

Proof. Suppose that $\theta(\Gamma)$ is not discrete. Then we can pick a net $(\mathbf{x}_i) \subseteq \Gamma$ such that the net $(\theta(\mathbf{x}_i))$ converges to some $\mathbf{a} \notin \theta(\Gamma)$. Choose a compact open subgroup $V \leqslant H'$. There exists an ordinal $\mathbb{L}$, such that $\theta(\mathbf{x}_i) \in \mathbf{a}V$ and, consequently, $\mathbf{x}_i \in \theta^{-1}(\mathbf{a}V)$

for all $i \geqslant \mathbb{L}$. Notice that $\theta^{-1}(\mathbf{a}V)$ is compact because $K\backslash\theta^{-1}(\mathbf{a}V) \cong \mathbf{a}V \cap \theta(H)$ and both K and $\mathbf{a}V \cap \theta(H)$ are compact. As every net in a compact space has a convergent subnet, we can find a subnet $(\mathbf{y}_j)$ of $(\mathbf{x}_i)$ which converges in $\theta^{-1}(\mathbf{a}V)$. But as $(\mathbf{x}_i) \subseteq \Gamma$ by assumption, and Γ is discrete, the net $\mathbf{y}_j$ is eventually constant, i.e., there exists an ordinal $\mathbb{M}$, such that $\mathbf{y}_j = \mathbf{y}_{\mathbb{M}}$ for all $j \geqslant \mathbb{M}$. This yields a contradiction to our assumption $\mathbf{a} \notin \theta(\Gamma)$:

$$\mathbf{a} = \lim_{i \to \infty} \theta(\mathbf{x}_i) = \lim_{i \to \infty} \theta(\mathbf{y}_j) = \theta(\mathbf{y}_{\mathbb{M}}) \in \theta(\Gamma).$$

Let us move on to cocompactness of $\theta(\Gamma)$. We have a continuous map

$$\varphi : \theta(\Gamma)\backslash H' \to \theta(H)\backslash H'.$$

For every $\theta(H)\backslash\mathbf{h} \in \theta(H)\backslash H'$, the fibre $\varphi^{-1}(\theta(H)\mathbf{h}) = \{\theta(\Gamma)\tilde{\mathbf{h}} \mid \theta(H)\tilde{\mathbf{h}} = \theta(H)\mathbf{h}\}$. In other words, fibres of all points are homeomorphic to $\theta(\Gamma)\backslash\theta(H)$ which is compact as θ is continuous. Thus, we have a fibration (or a fibre bundle)

$$\theta(\Gamma)\backslash\theta(H) \dashrightarrow \theta(\Gamma)\backslash H' \xrightarrow{\varphi} \theta(H)\backslash H' ,$$

whose base $\theta(H)\backslash H'$ and whose fibres $\theta(\Gamma)\backslash\theta(H)$ are both compact. To finish the proof, it is enough to show that the map φ is proper. Choose an open cover $\{U_i\}$ for $\theta(H)\backslash H'$. Let $\varphi_i : \varphi^{-1}(U_i) \to U_i$. This map (up to homeomorphism) is equal to the projection $p_i : U_i \times \theta(\Gamma)\backslash\theta(H) \to U_i$, which is a proper map [8, I.10.2]. Thus, φ is also proper [8, I.10.2].

□

With notation as before, we make the following observation:

Lemma 7.3.14. *If Γ is a cocompact lattice in $\widehat{G}$, then $\Gamma \cap C = \{1_{\widehat{G}}\}$.*

Proof. First note that $\Gamma \cap C$ is finite since Γ is discrete and C is compact and first countable. Since $\widehat{U}$ is a Sylow pro-p subgroup of $\widehat{G}$, then $C \coloneqq \bigcap_{\mathbf{g} \in \widehat{G}} \mathbf{g}\widehat{U}\mathbf{g}^{-1}$ is the intersection of Sylow p-subgroups of $\widehat{G}$. Thus, every finite order element in $\Gamma \cap C$

must have order p^k, for some k. But $\widehat{G}$ is p-well-behaved, in particular, cocompact lattices do not contain elements of order p. It follows that $\Gamma \cap C = \{1_{\widehat{G}}\}$. $\square$

Corollary 7.3.15. *If $\Gamma \leqslant \widehat{G}$ is a cocompact lattice, then $\pi(\Gamma) \cong \Gamma$. In particular, the cocompact lattice $\pi(\Gamma) \leqslant \overline{G}$ contains no elements of order p.*

Proof. As before let $\pi : \widehat{G} \to \overline{G}$. By Lemma 7.3.6 $\widehat{G}/C \cong \overline{G}$ and since $\Gamma \cap C = \{1_{\widehat{G}}\}$, it follows that $\Gamma \cong \pi(\Gamma)$. Since cocompact lattices in $\widehat{G}$ do not contain elements of order p, the same is true for their isomorphic images. $\square$

Our last observation tackles the *pulling* of cocompact lattices from $\overline{G}$ to $\widehat{G}$.

Proposition 7.3.16. *Recall that G denotes the minimal Kac-Moody group of rank 2. Let $\Gamma \leqslant G$ be a cocompact lattice in $\overline{G}$. Then Γ is also a cocompact lattice in $\widehat{G}$.*

Proof. Suppose $\Gamma \leqslant \widehat{G}$ is not discrete. Then there exists a sequence $\mathbf{x}_n \in \Gamma$ convergent to $\mathbf{a} \in \widehat{G}$, such that $\mathbf{a} \notin \Gamma$. Since $\pi(\mathbf{a}) = \lim_{n\to\infty} \pi(\mathbf{x}_n)$ and since $\Gamma \leqslant \overline{G}$ is discrete, $\pi(\mathbf{a}) \in \Gamma$ and the sequence $\pi(\mathbf{x}_n)$ is eventually equal to $\pi(\mathbf{a})$. Thus, there exists N, such that $\mathbf{x}_n \in \mathbf{a}C$ for all $n \geqslant N$. Moreover, $\mathbf{x}_n \in \mathbf{a}C \cap G$ for all $n \geqslant N$ since $\Gamma \subseteq G$.

We claim that the set $\mathbf{a}C \cap G$ has at most one element. Consider two of its elements $\mathbf{ag}$ and $\mathbf{ah}$, with $\mathbf{g}, \mathbf{h} \in C$. Then $(\mathbf{ag})^{-1}\mathbf{ah} = \mathbf{g}^{-1}\mathbf{h} \in C \cap G$. But $G \leqslant \overline{G}$ and $C \cap \overline{G} = \{1\}$. Thus, $C \cap G = \{1\}$, hence $\mathbf{g}^{-1}\mathbf{h} = 1$ and $|\mathbf{a}C \cap G| \leqslant 1$. It follows that $\mathbf{x}_n = \mathbf{x}_N$ for all $n \geqslant N$ and $\mathbf{a} = \mathbf{x}_N \in \Gamma$, a contradiction, showing that Γ is discrete in $\widehat{G}$.

Let

$$\varphi : \Gamma\backslash\widehat{G} \to \Gamma\backslash\overline{G}, \quad \Gamma\mathbf{g} \mapsto \Gamma\pi(\mathbf{g}).$$

Clearly φ is surjective. Observe that the fibers $\varphi^{-1}(\mathbf{g}) = \mathbf{g}C$. Thus, every fiber is homeomorphic to C and is compact. Thus, once again we have a fibration:

$$C \dashrightarrow \Gamma\backslash\widehat{G} \longrightarrow \Gamma\backslash\overline{G}.$$

Repeating the argument from Proposition 7.3.13 finishes the proof.

$\square$

7.3.2 Covolumes of cocompact lattices

In this section we follow Bass and Lubotzky with notation and conventions [2]. The theorem presented is original work and it appears in the joint paper of Inna Capdeboscq, Dmitriy Rumynin and the author of this thesis [10].

Defintion 7.3.17. [2, 1.5] Let H be a locally compact group acting on a set X with compact open stabilisers H_x, $x \in X$, and let $\Gamma \leqslant H$ be a discrete subgroup. The *covolume* of Γ is defined as

$$\nu ol(\Gamma\backslash\backslash X) := \sum_{[x]\in\Gamma\backslash X} \frac{1}{|\Gamma_x|},$$

where $\Gamma_x = H_x \cap \Gamma$.

Note that $\Gamma\backslash\backslash X$ is a notation which can be interpreted as a "weighted quotient". More precisely, one can think of $\Gamma\backslash\backslash X$ as the quotient $\Gamma\backslash X$ with the "weight" $1/|\Gamma_x|$ attached to each element Γx of $\Gamma\backslash X$ [2].

The covolume of Γ is finite if and only if Γ is a lattice (forcing H to be unimodular) and

$$\mu(H\backslash\backslash X) := \sum_{[x]\in H\backslash X} \frac{1}{\mu(H_x)} < \infty,$$

where μ is a right-invariant Haar measure on H. In this case, we can choose μ on H in such a way that (see [2, 1.5])

$$\nu ol(\Gamma\backslash\backslash X) = \mu_{\Gamma\backslash H}(\Gamma\backslash H).$$

We wish to consider covolumes of cocompact lattices in $\widehat{G}$ and $\overline{G}$. Recall that a rank 2 Kac-Moody group over a finite field $\mathbb{F}_q$, where $q = p^a$ for some prime p, has a building $\mathcal{BT}$ which is a $(q+1)$-regular tree. The set of vertices $\mathcal{BT}_0$ consists of conjugates of the parabolic subgroups P_1 and P_2 described in Section 7.3.1. Let x_i denote the vertex corresponding to P_i and $[x_i]$ its equivalence class under the action of G, $i = 1, 2$. Then

$$G\backslash\mathcal{BT}_0 = [x_1] \sqcup [x_2].$$

Both $\overline{G}$ and $\widehat{G}$ act on $\mathcal{BT}$. Abusing notation we also write $[x_i]$ for the $\overline{G}$ and $\widehat{G}$ equivalence classes of x_i.

Proposition 7.3.18. *It is possible to normalise the Haar measures $\widehat{\mu}$ on $\widehat{G}$ and $\overline{\mu}$ on $\overline{G}$ in such a way that*

$$\widehat{\mu}(\Gamma\backslash\widehat{G}) = \overline{\mu}(\Gamma\backslash\overline{G}) = \sum_{[x]\in\Gamma\backslash\mathcal{BT}_0} \frac{1}{|\Gamma_x|}$$

for any cocompact lattice $\Gamma \leqslant \widehat{G}$, where, by abuse of notation, $\widehat{\mu}$ and $\overline{\mu}$ are also the induced measures on $\Gamma\backslash\widehat{G}$ and $\Gamma\backslash\overline{G}$ correspondingly.

Proof. First note that as $\widehat{G}$ and $\overline{G}$ are both locally compact totally disconnected groups, they indeed admit left (and right) Haar measures.

Now, if there exist no cocompact lattices, the statement is tautologically true. So suppose $\Gamma \leqslant \widehat{G}$ is a cocompact lattice. Then $\pi(\Gamma) \cong \Gamma$ is a cocompact lattice in $\overline{G}$. Thus, the groups are unimodular [2]. The group $\overline{G}$ acts on $\mathcal{BT}$ and the stabilisers $\overline{G}_x$ for all $x \in \mathcal{BT}_0$ are compact open subgroups. In particular, $\overline{\mu}(\overline{G}_x) < \infty$. Thus,

$$\mu(\overline{G}\backslash\backslash\mathcal{BT}) = \sum_{[x]\in\overline{G}\backslash\mathcal{BT}_0} \frac{1}{\overline{\mu}(\overline{G}_x)} = \frac{1}{\overline{\mu}(\overline{G}_{x_1})} + \frac{1}{\overline{\mu}(\overline{G}_{x_2})} < \infty,$$

where x_1 and x_2 are representatives of the orbits of P_1 and P_2 in $\overline{G}$ respectively. It follows that $\overline{\mu}$ can be normalised so that

$$\overline{\mu}(\Gamma\backslash\overline{G}) = \sum_{[x]\in\pi(\Gamma)\backslash\mathcal{BT}_0} \frac{1}{|\pi(\Gamma)_x|} = \sum_{[x]\in\Gamma\backslash\mathcal{BT}_0} \frac{1}{|\Gamma_x|}.$$

Now consider the orbits of x_1 and x_2 under the action of $\widehat{G}$. Again, we have compact open stabilisers $\widehat{G}_x$, for every $x \in \mathcal{BT}_0$. By the same argument as above

$$\mu(\widehat{G}\backslash\backslash\mathcal{BT}) = \sum_{[x]\in\widehat{G}\backslash\mathcal{BT}_0} \frac{1}{\widehat{\mu}(\widehat{G}_x)} < \infty.$$

Consequently,

$$\widehat{\mu}(\Gamma\backslash\widehat{G}) = \sum_{[x]\in\Gamma\backslash\mathcal{BT}_0} \frac{1}{|\Gamma_x|} = \overline{\mu}(\Gamma\backslash\overline{G})$$

as required. □

We finish the discussion on lattices with the following theorem:

Theorem 7.3.19. *Let $\mathcal{A}$ be a symmetric 2×2 generalised Cartan matrix with all $|a_{ij}| \geqslant 2$. Let $\mathfrak{D}$ be a simply-connected root datum of type $\mathcal{A}$. The following statements hold for the corresponding (to $\mathfrak{D}$) locally pro-p-complete Kac-Moody group $\widehat{G}$ over the field of $q = p^a$ elements:*

1. *$\widehat{G}$ admits a cocompact lattice.*

2. *If $q \geqslant 514$, then there exist $\delta \in \{1, 2, 4\}$, such that*

$$\min\{\widehat{\mu}(\Gamma\backslash\widehat{G}) \mid \Gamma \textit{ is a cocompact lattice}\} = \frac{2}{(q+1)|Z(G)|\delta} .$$

Proof. In [12, Th. 1.1] Capdeboscq and Thomas classify, and, in particular, show existence of, edge-transitive cocompact lattices in the complete rank 2 Kac-Moody group $\overline{G}$ over $\mathbb{F}_q$ with a symmetric generalised Cartan matrix. Using Corollary 7.3.15 we can pull these lattices to $\widehat{G}$, thus proving Statement (1). To prove Statement (2) we use [12, Th. 1.3], where covolumes of lattices in $\overline{G}$ are computed. To obtain the covolumes in $\widehat{G}$, we apply Proposition 7.3.18. This finishes the proof of the theorem. □

Capdeboscq and Thomas also classify edge-transitive p-well-behaved cocompact lattices in $\overline{G}$ [12]. Now Corollary 7.3.15 and [12, Th. 1.3] together give us a classification of edge-transitive cocompact lattices in $\widehat{G}$.

7.4 Representations of Kac-Moody groups

All the material in this section is original, unless otherwise stated. It is taken from a joint paper by Dmitriy Rumynin and the author of this thesis [32]. The aim of this section is to study projective resolutions of smooth representations of groups with generalised (B, N)-pairs, as well as complete Kac-Moody associated to a root datum $\mathfrak{D}$ and a finite field $\mathbb{F}_q$ by constructing appropriate simplicial sets on which those groups act, in order to apply Theorem 5.3.1. The field $\mathbb{F}$ as in Chapters 4, 5 and 6 over which we consider smooth representations appears again. We put appropriate restrictions on it characteristic as we go along. If the field $\mathbb{K}$ appears, will be the field to which we have an associated minimal Kac-Moody group. This field is completely arbitrary and no restrictions on its characteristic apply.

7.4.1 Groups with a generalised (B, N)-pair

In Section 7.2 we defined groups with a (B, N)-pair structure. Now we want to look at a similar class. Following Iwahori [35], we propose the following definition:

Defintion 7.4.1. [35] A *generalised* (B, N)*-pair* on a group G is a triple (B, N, S) satisfying the following conditions:

(i) B and N are subgroups of G, $H = B \cap N$ is a normal subgroup of N.

(ii) $N/H = \Omega \ltimes W$, where Ω is a subgroup and W is a normal subgroup.

(iii) W is generated by the set S. The elements of S have the following properties:

   (iii.1) For any $\mathbf{t}$ in $\Omega \ltimes W$ and any $\mathbf{s} \in S$ we have $\dot{\mathbf{t}}B\dot{\mathbf{s}} \subset B\dot{\mathbf{t}}\dot{\mathbf{s}}B \cup B\dot{\mathbf{t}}B$ where $\dot{\mathbf{t}}$ and $\dot{\mathbf{s}}$ are elements of G lifting $\mathbf{t}$ and $\mathbf{s}$.

   (iii.2) $\mathbf{s}^2 = 1$ and $\dot{\mathbf{s}}B\dot{\mathbf{s}}^{-1} \neq B$ for all $\mathbf{s} \in S$.

(iv) $\mathbf{a}S\mathbf{a}^{-1} = S$ for all $\mathbf{a} \in \Omega$.

(v) $\dot{\mathbf{a}}B\dot{\mathbf{a}}^{-1} = B$ for all $\mathbf{a} \in \Omega$ and $B\dot{\mathbf{a}} \neq B$ for any $\mathbf{a} \in \Omega \backslash \{1\}$.

(vi) G is generated by B and N.

As usual W is called the Weyl group of G. Similarly to the case of groups with standard (B, N)-pairs (W, S) is a Coxeter system. We call $\Omega \ltimes W$ *the generalised Weyl group* of G.

Given a group G with a generalised (B, N)-pair, we can always find a smaller group G_0 inside G which has a (B, N)-pair. More precisely, define $G_0 := BWB$. Then the following statements hold:

Lemma 7.4.2. *[35]*

1. *G_0 is a normal subgroup of G and $G/G_0 \cong \Omega$.*

2. *(B, N_0) is a (B, N)-pair for G_0, where $N_0 = N \cap G_0$. The Weyl groups of G_0 and G are the same.*

3. *The automorphism of G_0 defined by conjugation by an element $\mathbf{g} \in G$ preserves the (B, N)-pair up to conjugacy in G_0, i.e., there exists $\mathbf{g}_0 \in G_0$, such that $\mathbf{g}B\mathbf{g}^{-1} = \mathbf{g}_0 B \mathbf{g}_0^{-1}$ and $\mathbf{g}N_0\mathbf{g}^{-1} = \mathbf{g}_0 N_0 \mathbf{g}_0^{-1}$.*

Let G be a group with a generalised (B, N)-pair and $G_0 \leqslant G$ be the subgroup admitting a (B, N)-pair. It follows that there exits a building $\mathcal{BT}$ for G_0, associated to its (B, N)-pair. By Part (3) of Lemma 7.4.2 $\mathcal{BT}$ admits a well-defined simplicial action of G. By definition of $\mathcal{BT}$ the fundamental apartment of $\mathcal{BT}$ is the Coxeter complex $\Sigma(W, S)$ of (W, S). Hence, there exists a colouring which identifies each vertex of the fundamental chamber with an element of S. Let G_1 be the subgroup of G that consists of all elements whose action on $\mathcal{BT}$ is type (colour)-preserving.

Lemma 7.4.3. *The following statements hold in the notations above.*

1. *G_1 is a normal subgroup of G containing G_0.*

2. *If K is the kernel of the G-action on $\mathcal{BT}$, then $G_1 = KG_0$.*

3. *(KB, N_1) is a (B, N)-pair for G_1, where $N_1 = N \cap G_1$.*

4. *The buildings and the Weyl groups of G_0 and G_1 are the same.*

5. *(KB, N) is a generalised (B, N)-pair for G with the same Weyl group (W, S).*

6. *If S is finite and the generalised Weyl group for the pair (KB, N) is $\Omega_1 \ltimes W$, then the constituent group Ω_1 is finite.*

Proof. Statement (1) is straightforward: $G_0 \subseteq G_1$ since the action of G_0 on $\mathcal{BT}$ is type-preserving. If $\mathbf{g} \in G$, $\mathbf{g}_1 \in G_1$, then the element $\mathbf{g}^{-1}\mathbf{g}_1\mathbf{g}$ is clearly in G_1 - if $\mathbf{g}$ changes the colouring, $\mathbf{g}^{-1}$ changes it back to the original one.

Since $K \leqslant G$ is the kernel of the G-action on $\mathcal{BT}$, then K acts trivially on $\mathcal{BT}$. Thus, all elements in KG_0 preserve the colouring and so $KG_0 \subseteq G_1$. Let $\mathbf{g} \in G_1$. By Lemma 7.4.2 (3) there exists a $\mathbf{g}_0 \in G_0$, such that $\mathbf{g}B\mathbf{g}^{-1} = \mathbf{g}_0 B\mathbf{g}_0^{-1}$ and $\mathbf{g}N_0\mathbf{g}^{-1} = \mathbf{g}_0 N_0\mathbf{g}_0^{-1}$. This means that $\mathbf{g}$ and $\mathbf{g}_0$ act on $\mathcal{BT}$ in the same way. Hence, we can write $\mathbf{g} = \mathbf{k}_0\mathbf{g}_0$, for some $\mathbf{k}_0 \in K$ and conclude $G_1 \subseteq KG_0$, finishing the proof.

Since $(B, N \cap G_0)$ is a (B, N)-pair for G_0 by Lemma 7.4.2 (2) and $G = KG_0$, Part (3) follows.

Let Σ be an apartment of $\mathcal{BT}$ and let (W, S) denote the Weyl group of G and G_0 (the Weyl groups of G and G_0 coincide by Lemma 7.4.2). By definition W consists of the type-preserving automorphisms of Σ. Since G_1 also acts by preserving the colouring, and $G_1 = KG_0$, where K acts trivially, it follows that the Weyl groups of G_0 and G_1 are the same. However, as the fundamental apartment of the building of G_1 is the Coxeter complex of its Weyl group, the statement follows.

The proof of the first part of (5) is a straightforward consequence of the fact that (B, N, S) is a generalised (B, N)-pair for G and the definitions of G_0 and G_1. Since the Weyl groups of G and G_1 are the same, the only difference between the two generalised (B, N)-pair structures of G is the group Ω in the definition of the generalised Weyl group. In the case of the generalised (B, N)-pair (KB, N, S) this is $N/H \cong \Omega_1 \ltimes W$, where $\Omega_1 \cong G/G_1$.

To prove (6), consider two elements $\mathbf{g}, \mathbf{h} \in G$ changing the colouring in the same way. Then the element $\mathbf{g}\mathbf{h}^{-1}$ does not change the colouring and hence $\mathbf{g}\mathbf{h}^{-1} \in G_1$. In other words, we have an injective map:

$$\Omega_1 \cong G/G_1 \longrightarrow \mathrm{S}_n,$$

where $n = dim(\mathcal{BT}) = |S|$. $\square$

Following we make the following definition:

Defintion 7.4.4. [44] Let (W, S) be a Coxeter system. The *presentation diagram* $\Gamma(W, S)$ is a graph with vertex set $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ and edges labelled by m_{ij} if $m_{ij} < \infty$, where m_{ij} is the order of $\mathbf{s}_i\mathbf{s}_j$ in W.

Note that the connected components of $\Gamma(W, S)$ correspond to the special subgroups of W which appear as factors in its free product decomposition [44]. Thus, we propose the following definition:

Defintion 7.4.5. A *connected component of* (W, S) is a subgroup (W', S'), where $W' \leqslant W$, $S' \subseteq S$, such that (W', S') is a Coxeter system whose presentation diagram $\Gamma(W', S')$ is a connected component of the presentation diagram $\Gamma(W, S)$ of (W, S). If the connected component (W', S') is a finite Coxeter group, we say that it is *a connected component of finite type.*

Now let us go back to our setting of a group G with a generalised (B, N)-pair (B, N, S). Let W be the Weyl group of G and G_0. We know that (W, S) is a Coxeter system. Suppose that the set S is finite and let $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. Let K be the kernel of the action of G on the building $\mathcal{BT}$ of G_0 (coming from the (B, N)-pair of G_0). Even though G acts on $\mathcal{BT}$, the building $\mathcal{BT}$ has the downside that it does not have to be contractible or to admit a contractible geometric realisation. In order to apply our results from Chapter 5 we need $|\mathcal{BT}|$ to be contractible. Thus, we now proceed to define a new simplicial set, $\mathcal{D}_\bullet$, on which G acts, such that $|\mathcal{D}|$ is contractible. Let $\overline{G} = G/K$.

Defintion 7.4.6. (cf. [50]) Suppose that (W, S) splits into connected components (W_i, S_i), $i = 1, \ldots, l$, with $W = W_1 \times \ldots \times W_l$. Let P be a parabolic subgroup of G and $\overline{P} \coloneqq P/K$. We call the pair $H \leqslant P$ a *marked parabolic* of G if $\overline{P} = \overline{P_1} \times \ldots \times \overline{P_l}$ is a parabolic of finite type and $\overline{H} = \overline{H_1} \times \ldots \times \overline{H_l}$, where $\overline{H_i} = \{1\}$ if $P_i \neq W_i$, and $\overline{H_i}$ is a Borel subgroup if $P_i = W_i$.

Let $\mathcal{S}ph(S)$ denote the set of all spherical subsets of S (i.e., $J \subseteq S$, such that the special subgroup $W_J = \langle J \rangle$ of W is finite). Using $\mathcal{S}ph(S)$, we would like to define a partially ordered set $\mathcal{P}(G)$.

Defintion 7.4.7. (cf. [50]) With notation as above, there is a set $\mathcal{P}(G)$ defined as follows:

1. If (W, S) has no connected components of finite type, then $\mathcal{P}(G)$ is the set of all proper spherical parabolic subgroups $P_J \leqslant G$.

2. If (W, S) splits into connected components with some components of finite type, then $\mathcal{P}(G)$ is the set of all marked parabolics of G.

Note that in either case the set $\mathcal{P}(G)$ is partially ordered. In case (1) $P_{J_0} \preceq P_{J_1}$ if $J_0 \subseteq J_1$, for some $J_0, J_1 \in \mathcal{S}ph(S)$. In case (2) $(H_{J_0} \leqslant P_{J_0}) \preceq (H_{J_1} \leqslant P_{J_1})$ if $H_{J_1} \subseteq P_{J_0} \subseteq P_{J_1}$. We write $P_{J_0} \preceq_{H_{J_0}} P_{J_1}$.

Defintion 7.4.8. (cf. [50]) The *Davis building* of a group G with a generalised (B, N)-pair (B, N, S) is the simplicial set $\mathcal{D}_\bullet = (\mathcal{D}_n)$, where $\mathcal{D}_n$ the set of all chains of length $n + 1$ in $\mathcal{P}(G)$, i.e.,

$$\mathcal{D}_n = \{P_{J_0} \preceq P_{J_1} \preceq ... \preceq P_{J_n}\}.$$

We also have the sets $\mathcal{D}_{(n)}$ of proper chains:

$$\mathcal{D}_{(n)} = \{P_{J_0} \prec P_{J_1} \prec ... \prec P_{J_n}\}.$$

The definition remains the same if (W, S) has components of finite type, but we substitute the parabolics P_{J_i} in the definition of n-simplices above with marked parabolics $H_{J_i} \leqslant P_{J_i}$. The *geometric realisation* $|\mathcal{D}|$ of $\mathcal{D}_\bullet$ is the geometric realisation of the partially ordered set $\mathcal{P}(G)$.

Since all parabolics are conjugates of standard parabolics P_J, some $J \subseteq S$, then an arbitrary n-simplex in $\mathcal{D}_\bullet$ will look like

$$\mathbf{g}_0 P_{J_0} \mathbf{g}_0^{-1} \preceq \ldots \preceq \mathbf{g}_n P_{J_n} \mathbf{g}_n^{-1},$$

with $P_{J_0} \preceq P_{J_1}$ and $\mathbf{g}_0^{-1}\mathbf{g}_1 \in P_{J_1}$. Using the bijection [7, V.2B, Corollary]

$$\{\mathbf{g}P, \mathbf{g} \in G_0, P \text{ parabolic subgroup}\} \longleftrightarrow \{\mathbf{g}P\mathbf{g}^{-1}, \mathbf{g} \in G_0, P \text{ parabolic subgroup}\},$$

we can rewrite the chain above as

$$\mathbf{g}_0 P_{J_0} \preceq \ldots \preceq \mathbf{g}_n P_{J_n},$$

with $P_{J_0} \preceq P_{J_1}$ and $\mathbf{g}_0^{-1}\mathbf{g}_1 \in P_{J_1}$. The marked case is similar, we just substitute parabolics with marked parabolics.

The group G acts on $\mathcal{P}(G)$ and, hence, on $\mathcal{D}_\bullet$, by conjugation. We can explicitly compute the stabilises of simplices under this action.

Lemma 7.4.9. *Let $x = [\mathbf{g}_0 P_{J_0} \preceq \ldots \preceq \mathbf{g}_n P_{J_n}] \in \mathcal{D}_n$. The stabiliser G_x is equal to $\mathbf{g}_0 B\Omega_x W_{J_0} B\mathbf{g}_0^{-1}$, where $\Omega_x = \bigcap_{i=0}^n \Omega_{J_i}$ and Ω_J is the stabiliser of J. In the marked case, $B = H_{J_0}$.*

Proof. By the definition of the partial order, for every $i \leqslant n$, there exists an element $\mathbf{p}_i \in P_{J_i}$ with $\mathbf{g}_{i-1}^{-1}\mathbf{g}_i = \mathbf{p}_i$. Recursively we can write $\mathbf{g}_i = \mathbf{g}_0\mathbf{p}_1 \ldots \mathbf{p}_i$. Hence

$$(G_0)_{\mathbf{g}_i P_{J_i}} = \mathbf{g}_i P_{J_i} \mathbf{g}_i^{-1} = \mathbf{g}_0 \mathbf{p}_1 \ldots \mathbf{p}_i P_{J_i} \mathbf{p}_i^{-1} \ldots \mathbf{p}_1^{-1} \mathbf{g}_0^{-1} = \mathbf{g}_0 P_{J_i} \mathbf{g}_0^{-1},$$

since $P_{J_k} \subseteq P_{J_i}$ for all $k \leqslant i$. This allows us to compute the stabiliser in G_0:

$$(G_0)_x = \bigcap_{i=0}^{n} (G_0)_{P_{J_i}} = \bigcap_{i=0}^{n} \mathbf{g}_0 P_{J_i} \mathbf{g}_0^{-1} = \mathbf{g}_0 P_{J_0} \mathbf{g}_0^{-1}.$$

Now, we move on to G_x. For every subgroup P of G containing B, there exists a unique subset $J \subseteq S$ and a unique subgroup Ω' of Ω, such that $P = B\Omega' W_J B$ [35]. The subgroup $\mathbf{g}_0^{-1} G_x \mathbf{g}_0$ contains B, hence, it is one of these subgroups. Moreover, as we know its intersection with G_0, we can conclude that

$$\mathbf{g}_0^{-1} G_x \mathbf{g}_0 = G_{\mathbf{g}_0^{-1}\cdot x} = B\Omega' W_{J_0} B = \bigcup_{\mathbf{u} \in \Omega'} B\dot{\mathbf{u}} W_{J_0} B$$

for some subgroup $\Omega' \leqslant \Omega$. Clearly, $\mathbf{u} \in \Omega'$ if and only if its lifting $\dot{\mathbf{u}}$ stabilises all cosets in $\mathbf{g}_0^{-1} \cdot x$, i.e., all P_{J_i}. Thus, $\Omega' = \bigcap_{i=0}^{n} \Omega_{J_i}$. $\square$

7.4.2 Topological groups of Kac-Moody type and their projective resolutions

Using our knowledge of generalised (B, N)-pairs, we propose the following definition.

Defintion 7.4.10. A topological group G is *a topological group of Kac-Moody type* if it admits a generalised (B, N)-pair (B, N, S), such that:

(1) G is a locally compact totally disconnected topological group.

(2) The set S is finite.

(3) The subgroup B is open in G.

(4) The subgroup B contains the kernel K of the G-action on the building associated to the subgroup G_0.

(5) If $J \subseteq S$ is a spherical subset, then P_J/K is compact.

Now we are ready for the main result of this section.

Theorem 7.4.11. *A topological group G of Kac-Moody type acts continuously on its Davis building $\mathcal{D}_\bullet$. Moreover, the stabiliser of each $x \in \mathcal{D}_n$ is compact modulo the action kernel K.*

Proof. The continuity of action is equivalent to all stabilisers G_x being open. This follows from Lemma 7.4.9 and B being open.

Since B contains the kernel K, $G_1 = G_0$ by Lemma 7.4.3. Moreover, Lemma 7.4.3 implies that the subgroup Ω is finite. As $\mathcal{D}_\bullet$ incorporates only spherical parabolic subgroups of G_0, each stabiliser G_x is union of finitely many double cosets $B\dot{\mathbf{w}}B$. Since K is normal, $(B\dot{\mathbf{w}}B)/K$ is the quotient topological space of $B/K \times B/K$. Thus, each double coset $B\dot{\mathbf{w}}B$ is compact modulo K and so is G_x. $\square$

We are going to repeatedly use this property of the Davis building, which was proved by Davis:

Theorem 7.4.12. *[20] If the set S is finite, the geometric realisation $|\mathcal{D}|$ of the Davis building is a $CAT(0)$ metric space with a piecewise Euclidean structure.*

More precisely, $|\mathcal{D}|$ is a unique geodesic space, and thus is contractible. Thus, the Davis building is the right simplicial set to consider if we would like to investigate projective resolutions of smooth representations of topological groups of Kac-Moody type. In particular, a topological group of Kac-Moody type G is locally compact totally disconnected, so we can look at its category of smooth representations $\mathcal{M}(G)$ and $\mathcal{M}_A(G)$, where $A \leqslant G$ is a closed central subgroup of G. With all the knowledge we gained about G and its Davis building $\mathcal{D}_\bullet$ we can now apply Theorem 5.3.1 in the current setting.

Corollary 7.4.13. *Let G be a topological group of Kac-Moody type, A its central closed subgroup, such that B/A is compact. The localisation functor for the category of A-semisimple G-representations over a field $\mathbb{F}$*

$$\mathcal{M}_A(G) \xrightarrow{\cong} \mathrm{Csh}_{G,A}(\mathcal{D}_\bullet)[\Sigma_A^{-1}]$$

is an equivalence of categories. Let $C \leqslant G$ be a compact subgroup, such that $\mathbb{F}$ is C-ordinary. Suppose further that the field $\mathbb{F}$ is G_x/A-ordinary for any $x \in \mathcal{D}_\bullet$, then

$$\mathrm{proj.\,dim}(\mathcal{M}_A(G)) \leqslant \sup_{J \in Sph(S)} |J|$$

where $|J|$ denotes the cardinality of J.

Proof. The kernel K contains any central subgroup, so certainly $A \subseteq K$. Moreover, B/A is compact if and only if B/K is compact. As B is a parabolic of finite type, this is true by the axioms of a topological group of Kac-Moody type. Since there exists a compact subgroup $C \leqslant G$, such that $\mathbb{F}$ is C-ordinary, we can define the Hecke algebra $\mathcal{H}(G, \mathbb{F}, \mu_C)$. Thus, by Proposition 5.1.1 and Theorem 4.5.2, the category $\mathcal{M}_A(G)$ has enough projectives. By Theorem 7.4.11 G acts on its Davis building with open compact modulo K, and thus A, stabilisers. By Theorem 7.4.12 $|\mathcal{D}|$ is contractible. So all conditions of Theorem 5.3.1 and Theorem 6.2.5 are satisfied and the result follows. $\square$

We finish this section with another observation about the class of groups we have introduced.

Theorem 7.4.14. *A topological group of Kac-Moody type G with compact B is unimodular.*

Proof. We can use the compact open subgroup B in Proposition 4.3.4 to compute the modular function. In particular, $\Delta(\mathbf{x}) = 1$ for all $\mathbf{x} \in B$. Part (5) of the definition of a generalised (B, N)-pair ensures that $\Delta(\dot{\mathbf{a}}) = 1$ for all $\mathbf{a} \in \Omega$. If $\mathbf{s} \in S$, then $\dot{\mathbf{s}}^{-1}B\dot{\mathbf{s}} = \dot{\mathbf{s}}B\dot{\mathbf{s}}^{-1}$, so again $\Delta(\dot{\mathbf{s}}) = 1$. The theorem follows because B, $\dot{S}$ and $\dot{\Omega}$ generate G. $\square$

7.4.3 Projective resolutions for complete Kac-Moody groups

Now let $G \coloneqq G_{\mathfrak{D}}(\mathbb{K})$ be a minimal Kac-Moody group over a field $\mathbb{K}$, $\mathfrak{D}$ its root datum of type $\mathcal{A}$ and $\mathcal{A}$ the generalised Cartan matrix of size $n \times n$. As discussed in Section 7.2.1, minimal Kac-Moody groups allow (B, N)-pair structure and thus have an associated building $\mathcal{BT}$ on which they act. However, neither $\mathcal{BT}$, nor its geometric realisation has to be a contractible space. Thus, similarly to Section 7.4.2, we define a Davis building $\mathcal{D}_\bullet$ for minimal Kac-Moody groups and use the property that $|\mathcal{D}|$ is contractible in order to apply our results from Chapter 5. We begin by defining $\mathcal{D}_\bullet$.

Defintion 7.4.15. Let $\mathcal{A}$ be an $n \times n$ generalised Cartan matrix. Suppose $\mathcal{A}$ is block diagonal, i.e., $\mathcal{A} = \mathcal{A}_1 \boxplus \mathcal{A}_2 \ldots \boxplus \mathcal{A}_l$, with $\mathcal{A}_i$ generalised Cartan matrices of size $n_i \times n_i$ and $\sum_{i=1}^{l} n_i = n$. Let $\mathfrak{D}_i$ be the root datum corresponding to $\mathcal{A}_i$, $i = 1, .., l$. Then $G_{\mathfrak{D}} = G_{\mathfrak{D}_1}(\mathbb{K}) \times \ldots \times G_{\mathfrak{D}_l}(\mathbb{K})$. We call $\mathcal{A}_i$ and $G_{\mathfrak{D}_i}(\mathbb{K})$ the *connected components* of $\mathcal{A}$ and G respectively. If $\mathcal{A}_i$ is a generalised Cartan matrix of finite type, we say that $\mathcal{A}_i$ and $G_{\mathfrak{D}_i}(\mathbb{K})$ are *connected components of finite type* of $\mathcal{A}$ and G respectively.

Let (W, S) denote the Weyl group of G with generating set S. Note that S is finite, and let $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. We have the same notions of special subset, special subgroup, spherical subset and parabolics of finite type as in Section 7.4.1. Let us describe the version of the Davis building for a minimal Kac-Moody group $G_{\mathfrak{D}}(\mathbb{K})$, where $\mathfrak{D}$ is a root datum of type $\mathcal{A}$ and $\mathbb{K}$ is a field. Again let K denote the kernel of the action of G on the building $\mathcal{BT}$ corresponding to the (B, N)-pair of G and let $\overline{G} = G/K$.

Defintion 7.4.16. [50] Suppose that G splits into connected components $G = G_1 \times \ldots \times G_l$. We call the pair $H \leqslant P$ a *marked parabolic* if $\overline{P} = \overline{P_1} \times \ldots \times \overline{P_l}$ is a parabolic of finite type and $\overline{H} = \overline{H_1} \times \ldots \times \overline{H_l}$, where $H_i = \{1\}$ if $P_i \neq G_i$, and H_i is a Borel subgroup if $P_i = G_i$, where $\overline{H} = H/K$, $\overline{P} = P/K$.

Using the above definitions for a connected component of G and marked parabolics, we obtain a partially ordered set $\mathcal{P}(G)$ defined in the same way as in Section 7.4.1. Thus:

Defintion 7.4.17. (cf. [50]) The *Davis building* of a minimal Kac-Moody group G is the simplicial set $\mathcal{D}_\bullet = (\mathcal{D}_n)$, where $\mathcal{D}_n$ the set of all chains of length $n+1$ in $\mathcal{P}(G)$. The *geometric realisation* $|\mathcal{D}|$ of $\mathcal{D}$ is the geometric realisation of the partially ordered set $\mathcal{P}(G)$.

Now let $\mathbb{K} \coloneqq \mathbb{F}_q$ be the field of $q = p^a$ elements, where p is a prime. Let $G = G_{\mathfrak{D}}(\mathbb{F}_q)$ be a minimal Kac-Moody group. Recall from Section 7.2.2 that in the situation when the minimal group is defined over a finite field, we can define completions of G which are locally compact totally disconnected groups. Denote the corresponding such complete groups by $\widetilde{G}$. The group $\widetilde{G}$ has a (B, N)-pair given by $(\widetilde{B}, N, S)$, where $\widetilde{B} = \widetilde{U} \rtimes H$, with H the torus of G. Thus, there is a corresponding building $\mathcal{BT}$ on which $\widetilde{G}$ acts. As mentioned above, neither $\mathcal{BT}$, nor its geometric realisation $\mathcal{BT}$ is contractible. However, the Davis building $\mathcal{D}_\bullet$ of G which we constructed above satisfies the property that $|\mathcal{D}|$ is contractible.

Let K be the kernel of the action of $\widetilde{G}$ on $\mathcal{BT}$. Then $K = \bigcap_{\mathbf{g} \in \widetilde{G}} \mathbf{g} B \mathbf{g}^{-1}$. Since $\widetilde{B}$ is open in $\widetilde{G}$, we can apply Lemma 7.4.9 and Theorem 7.4.11 to conclude that stabilisers of the action on $\widetilde{G}$ on $\mathcal{D}_\bullet$ are compact modulo K. Hence, we have the following consequences of Theorem 5.3.1 and Theorem 6.2.5:

Corollary 7.4.18. *Let $\widetilde{G}$ be as above and $\mathcal{D}_\bullet$ be its Davis building. Suppose there is a compact subgroup $C \leqslant \widetilde{G}$, such that $\mathbb{F}$ is C-ordinary. Further suppose that the field $\mathbb{F}$ is $\widetilde{G}_x/K$-ordinary for any $x \in \mathcal{D}_\bullet$, then*

$$\operatorname{proj.dim}(\mathcal{M}(\widetilde{G})) \leqslant \dim(\mathcal{D}_\bullet) \textit{ and } \mathcal{M}(\widetilde{G}) \cong Csh_{\widetilde{G}}(\mathcal{D}_\bullet)[\Sigma^{-1}].$$

Moreover, for A its central closed subgroup, such that $\widetilde{B}/A$ is compact, we have the equivalence

$$\mathcal{M}_A(\widetilde{G}) \xrightarrow{\cong} \mathrm{Csh}_{G,A}(\mathcal{D}_\bullet)[\Sigma_A^{-1}].$$

If the field $\mathbb{F}$ is $\widetilde{G}_x/A$-ordinary for any $x \in \mathcal{D}_\bullet$, then

$$\operatorname{proj.dim}(\mathcal{M}_A(\widetilde{G})) \leqslant \sup_{J \in \mathcal{S}ph(S)} |J|$$

where $|J|$ denotes the cardinality of J.

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